Stringy AdS_3/CFT_2

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based on arXiv pubs:

2012.00790 (T. Daniel Brennan, EJM)

2109.00065 (Bruno Balthazar, Amit Giveon, David Kutasov, EJM)

2109.11716 (EJM)

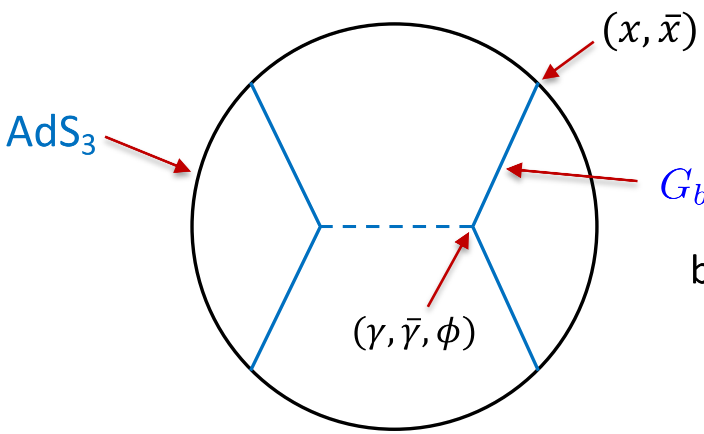
Intro/Outline

- Noncritical string theory has been a useful laboratory for exploring basic properties of string theory:
  - solvable examples of nonperturbative string theory
  - first example of gauge/gravity duality
  - rich structure of holographic RG flows, symmetries, phases, *etc.*

- Today we analyze noncritical string models related to *little string theory*:
  - review of AdS_3/CFT_2 limit of **F1**-**NS5** from the worldsheet
  - the long string sector and symmetric products $(\mathcal{M}^{\mathbf{p}})/S_{\mathbf{p}}$
  - the spectrum of $\mathcal{M}$ from asymptotics
  - walling off strong coupling
  - linear dilaton generalization and $T\overline{T}$

AdS_3 string theory

- In AdS_3 holography, spacetime CFT correlators can be calculated in perturbation theory via Feynman-Witten diagrams



$$G_{b\partial} = \left(e^{Q\phi}|\gamma - x|^2 + e^{-Q\phi}\right)^{-2j}$$

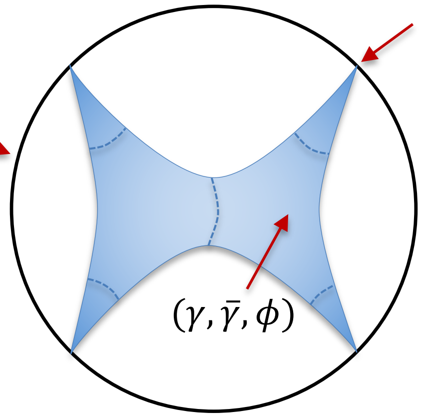
bulk-to-boundary propagator

$$ds^2 = d\phi^2 + e^{2Q\phi} d\gamma \, d\bar{\gamma}$$

AdS$_3$ string theory

- In AdS$_3$ string theory, perturbative spacetime CFT correlators can be calculated using worldsheet correlators of SL(2,R)$_k$ vertex ops $\Phi_j(x,\bar{x})$

current algebra level



$\phi,\gamma,\bar{\gamma}$	are worldsheet fields *e.g.* $\phi(z,\bar{z})$
$x,\bar{x}$	are c-number coherent state parameters

asymptotes to local bdy op

$$\Phi_j(x,\bar{x}) = \left(e^{Q\phi}|\gamma - x|^2 + e^{-Q\phi}\right)^{-2j} \sim e^{(j-1)Q\phi}\,\delta^2(\gamma - x) + \ldots$$

$G_{b\partial}$ solves the bulk wave eq w/scaling dim $h_j = \frac{-j(j-1)}{k}$

CFT symmetry from the worldsheet

- There are **worldsheet** representatives of the stress tensor supermultiplet of the **spacetime** CFT satisfying the usual OPE's

$$\mathcal{T}(x) = \int d^2z\ \underline{\mathfrak{t}(x)}\bar{\mathfrak{j}}(\bar{x})\Phi_1(x,\bar{x})$$ AdS$_3$ chiral graviton

$$\mathcal{G}^{\pm}(x) = \int d^2z\ \underline{\mathfrak{s}^{\pm}(x)}\bar{\mathfrak{j}}(\bar{x})\Phi_1(x,\bar{x})$$ chiral gravitino

$$\mathcal{J}_R(x) = \int d^2z\ \underline{\mathfrak{j}^3_{\rm su}(x)}\bar{\mathfrak{j}}(\bar{x})\Phi_1(x,\bar{x})$$ KK chiral gauge field mode

$$\mathcal{T}(x)\mathcal{T}(y) \sim \frac{3k\mathcal{I}}{(x-y)^4} + \frac{2\mathcal{T}(y)}{(x-y)^2} + \frac{\partial_y\mathcal{T}}{x-y}$$ *etc.*

where $\mathfrak{I}$ is a nontrivial **worldsheet** representative of the identity operator in **spacetime** (also the dilaton zero mode) w/ $\langle\mathfrak{I}\rangle \propto$ p = #**F1**'s

$$\mathcal{I}(x) = \int d^2z\ \underline{\mathfrak{j}(x)}\bar{\mathfrak{j}}(\bar{x})\Phi_1(x,\bar{x})$$

central term *i.e.* $\partial_x\mathcal{I} = \partial_{\bar{x}}\mathcal{I} = 0$
up to worldsheet BRST exact terms

AdS$_3$ string theory operators

- SL(2,R) operators:

coherent state parameter

local operator

Position basis

$$\Phi_j(x,\bar{x}) = \left(e^{Q\phi}|\gamma - x|^2 + e^{-Q\phi}\right)^{-2j} \;\sim\; e^{(j-1)Q\phi}\,\delta^2(\gamma - x) + \dots$$

Fourier basis

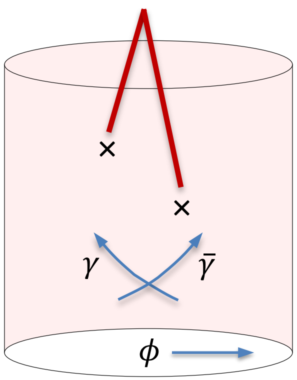
$$\Phi_{j;m,\bar{m}} \sim e^{(j-1)Q\phi}\,\gamma^{j+m}\bar{\gamma}^{j+\bar{m}} + \mathcal{R}(j,m)\,e^{-jQ\phi}\,\gamma^{m-j}\bar{\gamma}^{\bar{m}-j} + \dots$$

non-normalizable falloff

reflection coefficient

normalizable falloff

Local operator Insertions on ∂

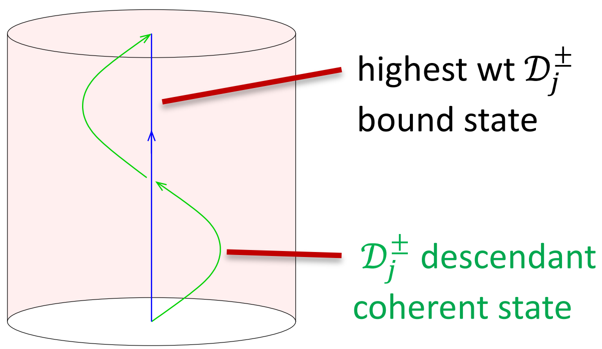


- The residues of poles in the reflection coefficient $\mathcal{R}$(j,m) correspond to **normalizable** wavefunctions (a sort of AdS$_3$ version of LSZ reduction) (Aharony-Giveon-Kutasov ‘04)
- These normalizable wavefunctions are unitary discrete series representations of SL(2,R) current algebra $\mathcal{D}_j^{\pm}$ for *½ < j < ½(k+1)* and describe string **states** bound to AdS$_3$
- (NB: *j<1* for *k<1*, so *e.g.* $\mathfrak{I}$ *is non-normalizable*)

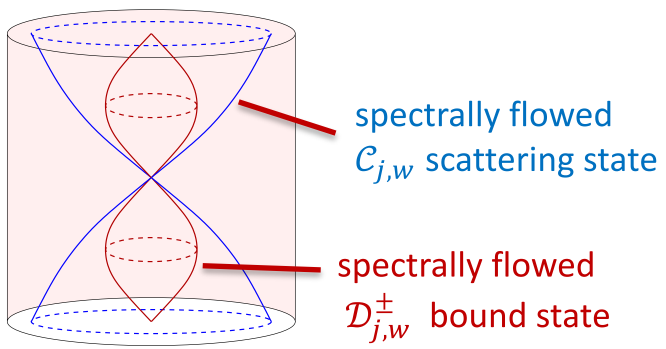
AdS$_3$ string theory states

- SL(2,R) spectral flow introduces *winding sectors* (Maldacena-Ooguri '00) including **scattering states**: continuous series $\mathcal{C}_{j,w}$ for $j = ½ + is$, $w>0$ = *long string sector*
- Semiclassical string motions on the SL(2,R)$_k$ group manifold take the following form (this is a bit of a cartoon for the small k regime we will be working in)

$$g(z,\bar{z}) = g_\ell(z)\, g_r(\bar{z}) \xrightarrow{\text{spectral flow}} g^{(w)}(z,\bar{z}) = e^{\frac{i}{2}wz\sigma_3}\, g(z,\bar{z})\, e^{\frac{i}{2}w\bar{z}\sigma_3}$$

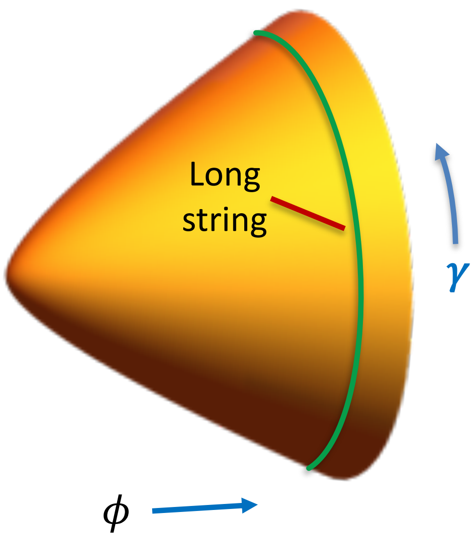


unexcited strings travel along AdS$_3$ geodesics



spectrally flowed solutions describe breathing modes

Long string effective theory



- Seiberg and Witten ('99) analyzed the spacetime EFT on a *long string* scattering state near the AdS boundary and found a sigma model on $R_\phi \times S^3 \times \Sigma_d$ where R_ϕ describes the radial direction in AdS_3; in the EFT, R_ϕ has a linear dilaton with slope

$$Q_\ell = (1-k)\sqrt{2/k}$$

Note: sign flip for k<1

- We will be interested in $\Sigma_d = T^4, T^2, \emptyset$

Long string effective theory

- Weakly coupled strings have a Fock space structure, suggesting that the spacetime CFT has asymptotic structure of the symmetric orbifold $(\mathcal{M}^p)/S_p$
- The component block $\mathcal{M}$ of the symmetric product should have the form of the SW theory $R_\phi \times S^3 \times \Sigma_d$; to match the central charge $c_{ST}=6kp$ of the spacetime CFT one needs $\mathcal{M}$ to have the spacetime central charge

$$c_{ST} = \underbrace{(1+3Q_\ell^2)}_{R_\phi} + \underbrace{(3-6/n)}_{S^3} + \underbrace{4\cdot\tfrac{1}{2}}_{fermions} + \underbrace{d\cdot\tfrac{3}{2}}_{T^d} = 6k$$

$$c_{WS} = (3+6/k) + (3-6/n) + 4\cdot\tfrac{1}{2} + d\cdot\tfrac{3}{2} = 15$$

$$\Rightarrow \quad Q_\ell = (1-k)\sqrt{2/k}$$

- For $k>1$ the asymptotic density of states of $(\mathcal{M}^p)/S_p$ is **strictly less than** the Cardy entropy w/central charge $c_{ST}=6kp$ (Kutasov-Seiberg '91)
- *There is more to the theory than perturbative string theory for* $k>1$, *i.e.* the spectrum of BTZ black holes; the spacetime CFT is *not* a symmetric product based on the Fock space of fundamental strings

A correspondence course

- Highly excited states in string theory are BH's at strong coupling, but become a string/brane gas at weak coupling (Horowitz-Polchinski '96)
- The figure of merit is the curvature at the horizon scale relative to $\ell_{\rm str}$
- For AdS_d, large BH's have horizon curvature set by the AdS scale $R_{\rm AdS}$
- The CFT dual to gravity is weakly coupled for $R_{\rm AdS} < \ell_{\rm str}$
- For AdS_3 w/NS fluxes (*i.e.* **F1**-**NS5**), $(R_{\rm AdS}/\ell_{\rm str})^2 = k$ where k is the level of the SL(2,R) current algebra that describes AdS_3 on the worldsheet
- There is a sharp correspondence transition at $k=1$ and *for $k<1$ the BTZ solution is not normalizable:* $S<S_{\rm BTZ}$ (Giveon-Kutasov-Rabinovici-Sever '05)

$$S = 2\pi\sqrt{c_{\rm eff}\mathcal{E}/3} \quad , \qquad c_{\rm eff} = c - h_{\rm min}/24$$

$h_{\rm min}>0$ for $k<1$ because
$\mathfrak{T}$ is not normalizable
($\Phi_1 \notin$ unitary range)

A correspondence course

- Below the correspondence transition, BTZ black holes are *absent* from the spectrum, whose asymptotics consists of the Hagedorn entropy of perturbative strings (Giveon-Kutasov-Rabinovici-Sever '05)

- The absence of a BTZ spectrum allows the possibility that for $k<1$ one has a spacetime CFT whose asymptotic spectrum **is** that of a sym product $(\mathcal{M}^{\mathbf{p}})/S_{\mathbf{p}}$

- The $k<1$ string theory is *noncritical* w/target $AdS_3 \times S^{3,\flat} = SL(2,R)_k \times SU(2)_n$ WZW model with $J^3_{\rm su}\bar{J}^3_{\rm su}$ deformation (Giveon-Kutasov-Pelc '99, Brennan-EJM '20)

$$c_{WS} = \underbrace{(3+6/k)}_{AdS_3} + \underbrace{(3-6/n)}_{S^3} + \underbrace{6\cdot\tfrac{1}{2}}_{fermions} = 15 \quad \longrightarrow \quad k = \frac{n}{n+1}$$

- The spacetime **F1**-**NS5** CFT with p **F1**'s has $c_{ST} = 6kp$

Long string effective theory

- The component block $\mathcal{M}$ of the symmetric product should have a central charge $c_{ST} = 6k = \frac{6n}{n+1}$ and indeed the central charge of $R_\phi x S^3$ is

$$c_{ST} = \underbrace{(1+3Q_\ell^2)}_{R_\phi} + \underbrace{(3-6/n)}_{S^3} + \underbrace{4\cdot\tfrac{1}{2}}_{fermions}$$

which determines $Q_\ell^2 = \frac{2}{n(n+1)}$

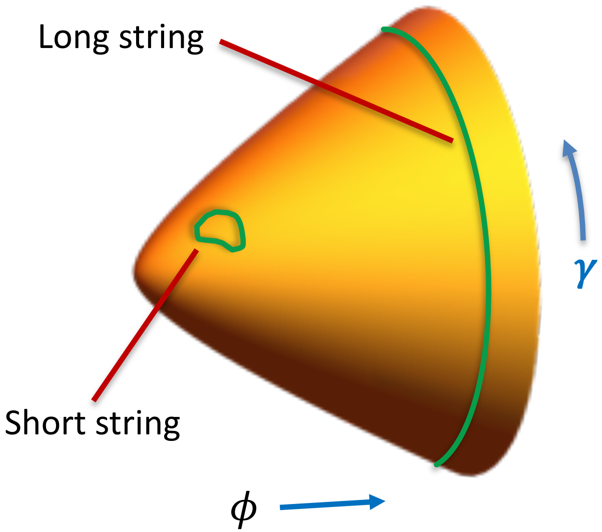
Q_ℓ = slope of R_ϕ linear dilaton

- Thus the symmetric orbifold CFT $(\mathcal{M}^p)/S_p$ with $\mathcal{M}=R_\phi x S^3$ reproduces the central charge of the spacetime CFT, and captures the asymptotic structure of the CFT Hilbert space. Could this be the correct dual theory?

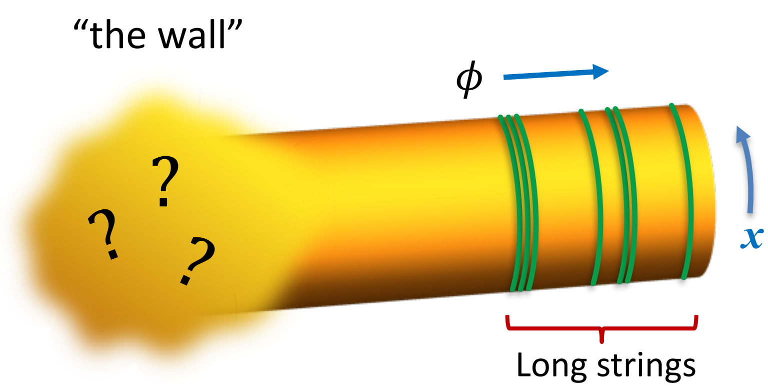
- Yes, if we can cure the strong coupling problem at the $\phi\rightarrow-\infty$ end of R_ϕ

The working hypothesis . . .

- Something similar to other linear dilaton theories is operating in the symmetric product (like the exponential wall in Liouville CFT) to keep strings out of strong coupling and mock up the AdS “cap” where long strings interact, reflect back toward the boundary, and/or unwind



Worldsheet description –
short and long strings in AdS_3



Spacetime description –
symmetric product CFT
of asymptotic long strings

A puzzle and its resolution

- There are (infinitely many) holomorphic operators such as $\partial\phi$ in the block theory $\mathcal{M}$ which are then holomorphic in the symmetric product $(\mathcal{M}^{\mathbf{p}})/S_{\mathbf{p}}$ as well; but the radial direction in AdS_3 is not described by free field theory.
What violates the apparent holomorphicity of these w=1 sector operators?

- The worldsheet knows the answer*: Computing $\bar{\mathcal{L}}_{-1}\mathcal{V}_{\partial\phi}$ using the worldsheet realization of these spacetime operators leads to a nonzero result. The operator $\partial\phi$ and therefore $\mathcal{V}_{\partial\phi}$ is non-normalizable, but $\bar{\mathcal{L}}_{-1}\mathcal{V}_{\partial\phi}$ sits on a **pole** in the reflection coefficient, and so underneath the naïve vanishing of $\bar{\partial}\partial\phi$ in the block theory is a normalizable residue. A piece of this normalizable operator is (the top component of) the $\mathbb{Z}_2$ BPS twist op

$$\boldsymbol{\tau} = e^{-\frac{n}{2}Q_\ell(\underline{\phi}+iY)}\boldsymbol{\sigma}_2 \quad , \qquad J^3_{su} = i\sqrt{\frac{n}{2}}\,\partial Y$$

where $\boldsymbol{\sigma}_2$ is the ground state $\mathbb{Z}_2$ twist operator.

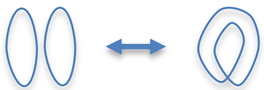
- The wall softly breaks the holomorphy of infinitely many higher spin currents

* because string theory is smarter than we are

Loss of freedom

- The symmetric product describes **non-interacting** long strings. But as in the critical dimension, there is a marginal $\mathbb{Z}_2$ twist operator $\boldsymbol{\tau}$ that generates string interactions. The worldsheet tells us that it is *necessarily turned on* in our case. This twist operator has a nontrivial profile in ϕ

$$\boldsymbol{\tau} = e^{-\frac{n}{2}Q_\ell(\phi+iY)}\boldsymbol{\sigma}_2$$

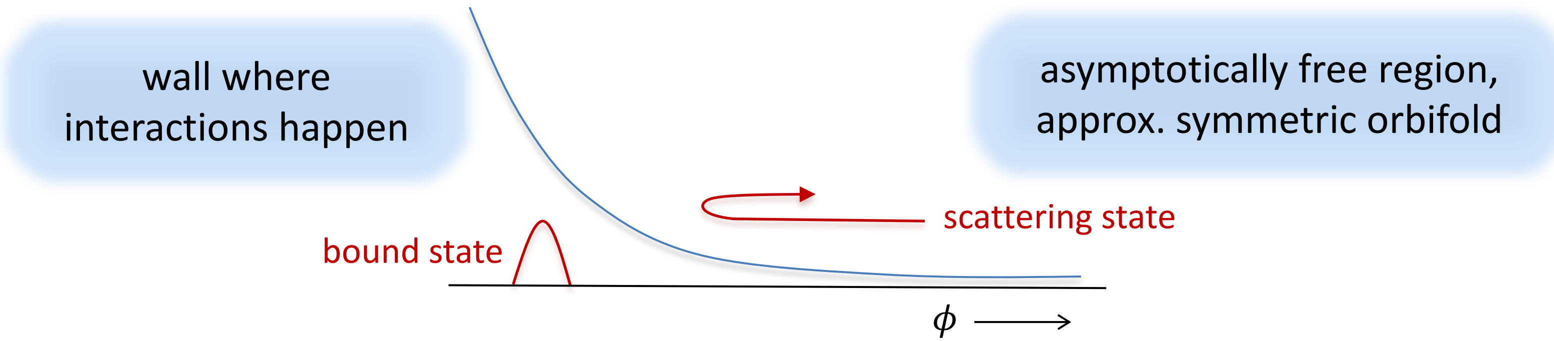


This operator allows strings to reconnect, but the interaction strength dies away at large positive ϕ, and grows at large negative ϕ. The symmetric product structure holds asymptotically – in a precise sense, the spacetime CFT is **asymptotically free**.

- The coefficient μ of this marginal operator does not parametrize a moduli space; rather, as in Liouville theory, it can be scaled away by a shift of ϕ – instead, it **sets a holographic scale** ϕ^* at which the interaction strength is O(1); and correlation functions obey (KPZ) scaling relations w.r.t. μ

Intuitions: Noncritical strings

- Thus the situation is very much akin to the $c = 1$ noncritical string models



- The analogue of the $c = 1$ tachyon is the spectrum of scattering states – the long string sector. This spectrum is completely captured by the symmetric product.

- A new feature is the plethora of bound states. It is perhaps not a surprise that this state space is a bit hidden from the Fock space of scattering states.

- The wall interaction sets a scale in ϕ; correlation functions obey (KPZ-like) scaling relations

LST holographic generalization

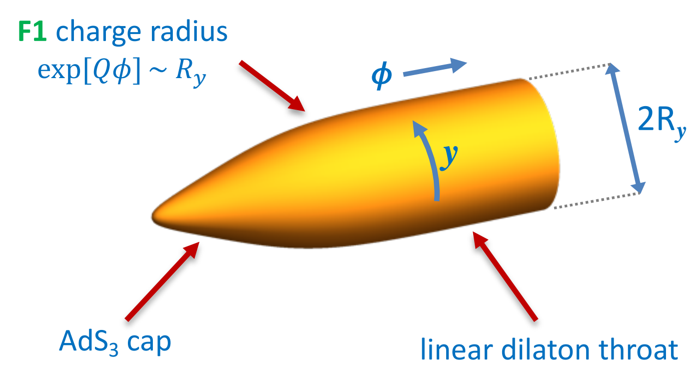
- AdS$_3$/CFT$_2$ string theory is an IR limit of *little string theory* holography
- One can introduce in the worldsheet theory a marginal deformation that changes the asymptotic geometry from AdS$_3$ to the linear dilaton asymptotics of little string theory
- One way to achieve this is via a gauged WZW model based on the group

$$\frac{\mathcal{G}}{\mathcal{H}} = \frac{SL(2,\mathbb{R})_k \times SU(2)^{\flat}_n \times \mathbb{R}_t \times \mathbb{S}^1_y}{U(1)_L \times U(1)_R}$$

where we gauge *e.g.* the left null current

$$\mathcal{J} = J^3_{\rm sl} + \sqrt{\tfrac{k}{n}}\, J^3_{\rm su} - R_y(\partial t + \partial y)$$

$$\sim \sqrt{k}\, e^{2Q\phi}\, \partial\gamma$$



Gauging mostly eliminates t,y ∈ $\mathbb{R}_t \times \mathbb{S}^1_y$ leaving $\gamma, \bar{\gamma}$ ∈ AdS$_3$

Gauging mostly eliminates $\gamma, \bar{\gamma}$ ∈ AdS$_3$ leaving t,y ∈ $\mathbb{R}_t \times \mathbb{S}^1_y$

LST holographic generalization

- For the scattering states, the energy $\mathcal{E}$ is determined by the worldsheet Virasoro constraint (*i.e.* string mass shell condition) to be

$$\frac{w_y R_y}{2}\Big(\mathcal{E}+\frac{k_y}{R_y}\Big)+\frac{\lambda}{2}\Big(\mathcal{E}^2-\frac{k_y^2}{R_y^2}\Big)=-\frac{j(j-1)}{k}+N_{\rm osc}+\Delta_{\mathbb{S}^3_\flat}$$

where $\mathcal{E}$ is the energy and λ is a deformation parameter (λ can be absorbed by a rescaling of $\mathcal{E}$ and R_y; the latter sets the physical scale)

- This relation is simply the Virasoro constraint of the worldsheet theory, but it is also the expression for the spectrum of $T\bar{T}$ deformed theories.

- When the CFT dual is a symmetric orbifold, this deformation is thought to indeed be dual to the $T\bar{T}$ deformation in the block of the sym product

$$\sum_i \det\big[\mathcal{T}_{ab}^{(i)}\big]$$

(Giveon-Itzhaki-Kutasov '17)

LST holographic generalization

- The *k<1* models are based on a symmetric product, and provide the appropriate arena for the application of the $T\bar{T}$ deformation to LST holography; generic spacetime CFT's dual to AdS_3 do **not** have a symmetric product structure, and then turning on the linear dilaton region is **not** a $T\bar{T}$ deformation (even if the spectrum of perturbative long strings has the same form); at best it describes a sector of the theory
- Even here, we must turn on the marginal $\mathbb{Z}_2$ twist deformation; this will lead to a modification of the $T\bar{T}$ structure at finite ϕ. Asymptotically though this structure holds, and that is sufficient to calculate the deformed spectrum of operators whose wavefunctions are concentrated in the asymptotic region, such as the scattering states.
- It would be interesting to understand how this proposal extends to the interacting theory, perturbed away from the symmetric orbifold by the marginal $\mathbb{Z}_2$ twist operator, which we need to match the worldsheet

Approximating black holes

- The internal space $\Sigma_d = T^d$ lies along the fivebranes. When it degenerates $(d \rightarrow 0)$ there are no transverse oscillations of the little string. BTZ black hole entropy is Hagedorn entropy of the little string; the loss of little string oscillator phase space explains the absence of BTZ black holes

- However, the spacetime CFT $(\mathcal{M}^{\mathbf{p}})/S_{\mathbf{p}}$ (deformed by the $\mathbb{Z}_2$ twist interaction) is not all that different for $\mathcal{M}=T^4$ and for $\mathcal{M}=R_\phi x S^3$ yet one is a model for black hole μstates while the other describes a gas of fundamental strings

- Note that T^4 describes the space *internal* to the fivebranes in the critical dim while $R_\phi x S^3$ describes the space *transverse* to the fivebranes for $k<1$. It is rare that the holographic description includes the holographic (radial) coord

- Nevertheless, the similarity gives us some hope that the $k<1$ models might usefully approximate black holes. How closely do the details match?

Approximating black holes

- High energy density of states $S = 2\pi\sqrt{c_{\rm eff}\mathcal{E}/3}$, $c_{\rm eff} = c - h_{\rm min}/24$ is saturated by the gas of long strings which has $h_{min} = \frac{p}{4n(n+1)}$ corresponding to each string being at the threshold of the continuum

- Because the lowest states have energies of order p, need to worry whether string perturbation theory is breaking down for the S-matrix of actual states

- Operators above are of the form (in the untwisted sector of $(\mathcal{M}^{\mathrm{p}})/S_{\mathrm{p}}$)

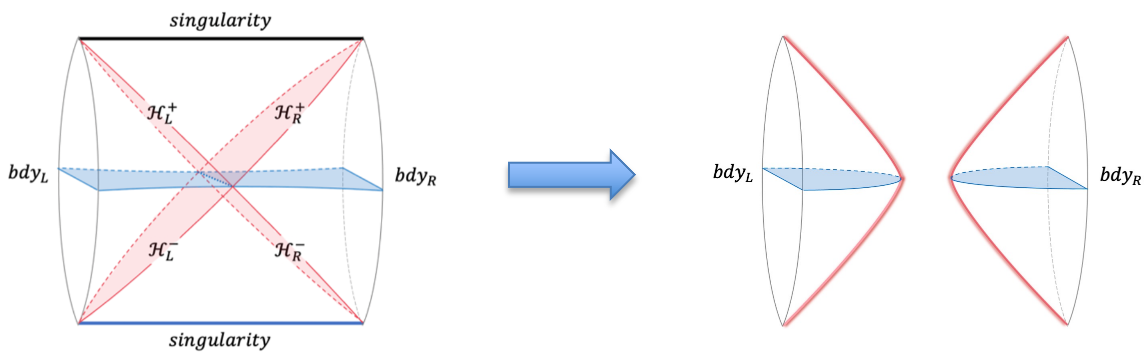
$$\sum_{i=1}^{p} \left(\mathbb{1}_1 \otimes \cdots \otimes \mathcal{V}_i \otimes \cdots \otimes \mathbb{1}_p \right)$$

  and do not correspond to states (because the identity op is non-normalizable).

- The worldsheet expands around an SL(2,R) invariant background with correlators of the above vertex ops, and is perhaps best thought of in terms of the Euclidean theory on $\mathbb{H}_3^+$ (where there are no states to worry about)

Approximating black holes

- The highly excited quasi-bound states that best approximate black holes will be supported at the wall where interactions are growing, and symprod twisted sectors involving long cycles (with length of order p) become important.
- The formation and decay of these states, and the extent to which this process approximates the formation and decay of black holes, will be interesting to study if we can develop the tools. Do they trap infalling energy for extended periods of time? Level repulsion? What is the chaos exponent? OTOC's? *etc.*
- However, there are fundamentally no horizons in these models



Summary

- AdS_3 string theory with $R_{\mathrm{AdS}} < \ell_{\mathrm{str}}$ provides an intriguing arena for holography
- BTZ black holes are absent from the asymptotic spectrum, which is instead populated with highly excited strings.
- The dual CFT is a deformation of the symmetric product of the SCFT on R_ϕ times (squashed) S^3 with a particular GSO projection
- Worldsheet string theory can be used to **derive** the symmetric product structure at asymptotically large ϕ
- The deformation is needed to introduce a wall that self-consistently keeps the theory weakly coupled
- Underlying these backgrounds is little string theory in a noncritical regime where strong coupling effects seem to be under better control