APCTP Focus Program: Exact result on irrelevant deformations of QFTs

Based on 2104.09529
Kyungsun Lee (GIST)
Piljin Yi (KIAS)
Junggi Yoon (APCTP)

$T\bar{T}$ Deformed Fermionic Theories Revisited

Junggi Yoon
APCTP

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apctp asia pacific center for theoretical physics

$T\bar{T}$-deformed Fermionic Theories Revisited

Kyungsun Lee, Piljin. Yi and JY
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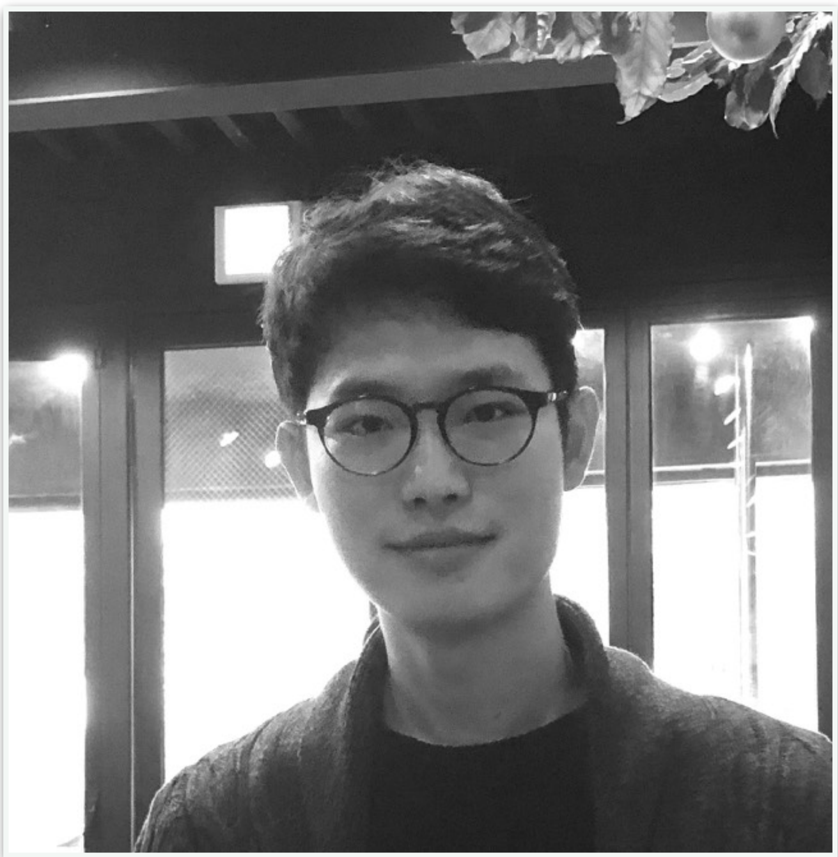
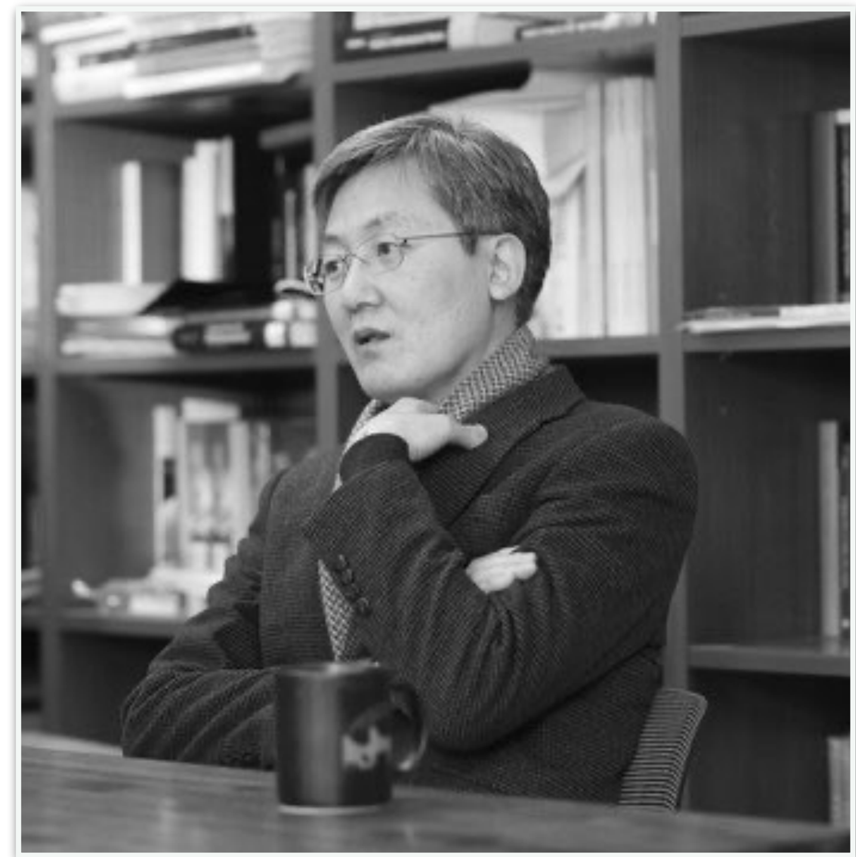


Kyungsun Lee
GIST



Piljin Yi
KIAS

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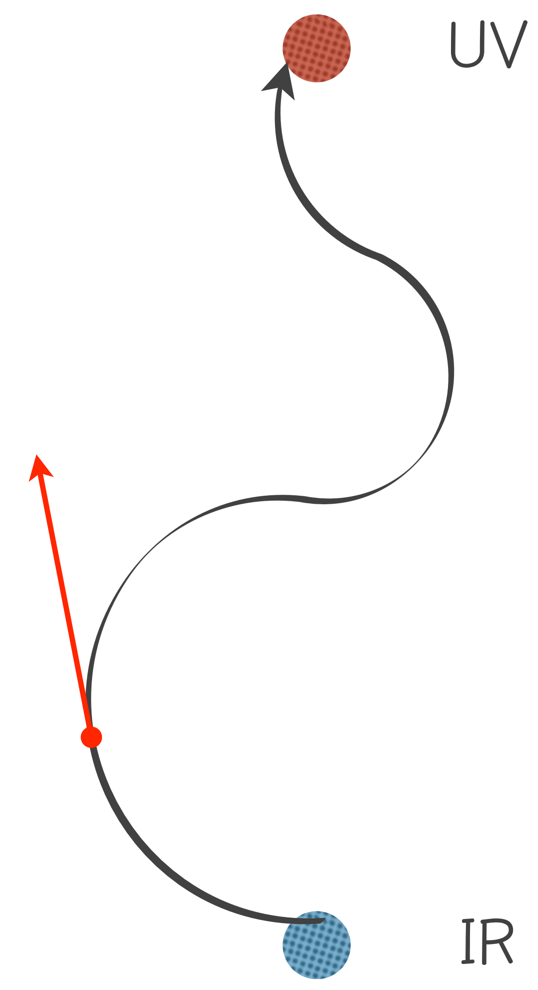
Motivation

$T\overline{T}$ Deformation

* Flow equation for $T\overline{T}$ deformation

$$\partial_\lambda \mathscr{L} = \frac{1}{2}\epsilon_{\mu\nu}\epsilon^{\rho\sigma}T^{\mu}{}_{\rho}T^{\nu}{}_{\sigma}$$



Features 1: Deformed Spectrum

✳ Universal formula for $T\overline{T}$ Deformed Spectrum

$$E_n(L,\lambda) = \frac{L}{2\lambda}\left[\sqrt{1+\frac{4\lambda}{L}E_n+\frac{4\lambda^2}{L^2}P_n^2}-1\right]$$

$$P_n(L,\lambda) = P_n(L)$$

E_n : Undeformed Energy

P_n : Undeformed Momentum

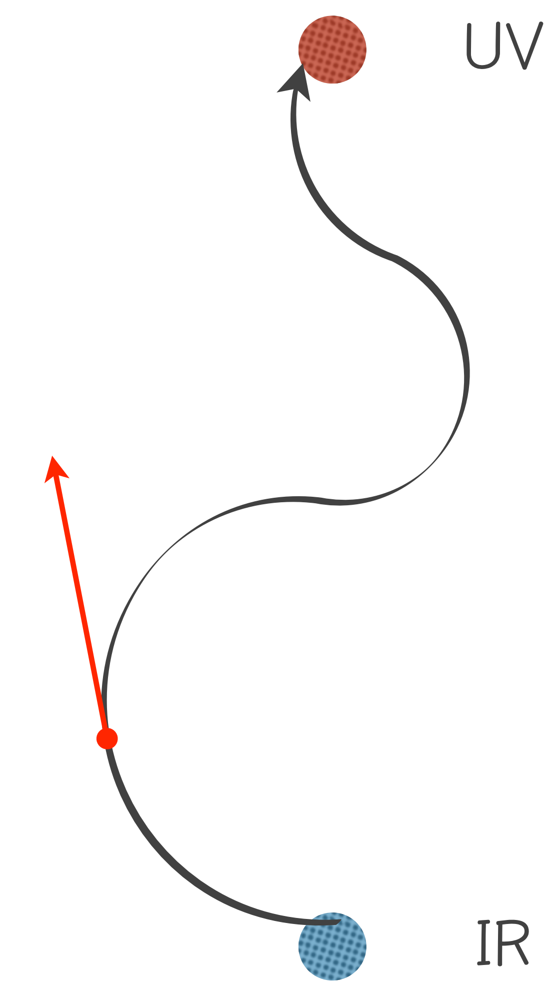


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Figure 2: $T\bar{T}$ Deformed Lagrangian

- Flow equation $\partial_\lambda \mathscr{L} = \frac{1}{2}\epsilon_{\mu\nu}\epsilon^{\rho\sigma}T^{\mu}{}_{\rho}T^{\nu}{}_{\sigma}$
  - ✓ EMT on RHS: Derivative of deformed Lagrangian w.r.t. field
  - ✓ This leads to differential equation of the Lagrangian
  - ✓ Can be solved perturbatively in principle. Mostly, we can find exact solutions.
  - ✓ Initial condition: $\mathscr{L}[\lambda = 0] = \mathscr{L}_{\text{undeformed}}$
  - ✓ e.g. Deformation of free scalar field: $\mathscr{L} = -\frac{1}{2\lambda}\left[\sqrt{1 + 2\lambda(-\dot{\phi}^2 + \phi'^2)} - 1\right]$
- This is related to Nambu-Goto action for 3D target space with "static gauge"
  - ✓ usually difficult to quantize the NG action
- Relation to dynamical coordinate transformation and 2D gravity

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Bridge between Lagrangian and Spectrum

- From the spectrum,

$$E_n(L,\lambda) = \frac{L}{2\lambda}\left[\sqrt{1+\frac{4\lambda}{L}E_n+\frac{4\lambda^2}{L^2}P_n^2}-1\right]$$

one might guess the deformed Hamiltonian as follow

$$H = \frac{L}{2\lambda}\left[\sqrt{1+\frac{4\lambda}{L}H_{(0)}+\frac{4\lambda^2}{L^2}P_{(0)}^2}-1\right]$$

- Is it too good to be true?

- "semi-classical" derivation of the above Hamiltonian from the string [Theisen, Jorjadze, 2020]: "gauge choice"

- What is the concrete relation between $T\bar{T}$ deformed Lagrangian and the above Hamiltonian?

$$\mathscr{L} = -\frac{1}{2\lambda}\left[\sqrt{1+2\lambda(-\dot{\phi}^2+\phi'^2)}-1\right] \quad \overset{?}{\Longleftrightarrow} \quad H = \frac{L}{2\lambda}\left[\sqrt{1+\frac{4\lambda}{L}H_{(0)}+\frac{4\lambda^2}{L^2}P_{(0)}^2}-1\right]$$

Take-home Messages

Equivalence of Two Hamiltonians

- From $T\bar{T}$ deformed Lagrangian
  - ✓ Flow equation: $\partial_\lambda \mathscr{L} = \frac{1}{2}\epsilon_{\mu\nu}\epsilon^{\rho\sigma}T^{\mu}{}_{\rho}T^{\nu}{}_{\sigma}$ $\quad \mathscr{L} = -\frac{1}{2\lambda}\left[\sqrt{1+2\lambda(-\dot{\phi}^2+\phi'^2)}-1\right]$
  - ✓ Legendre Transformation to Hamiltonian Density
    ➜ One integral for Hamiltonian
- From deformed Energy: $E_n(L,\lambda) = \frac{L}{2\lambda}\left[\sqrt{1+\frac{4\lambda}{L}E_n+\frac{4\lambda^2}{L^2}P_n^2}-1\right]$
  - ✓ might conjecture $H = \frac{L}{2\lambda}\left[\sqrt{1+\frac{4\lambda}{L}H_{(0)}+\frac{4\lambda^2}{L^2}P_{(0)}^2}-1\right]$
  - ✓ One integral inside square-root
    ➜ Many integrals in Taylor's expansion.

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Noether vs Metric Variation?

* Two ways to evaluate Energy-momentum tensor
  - ✓ Noether procedure: EMT is not symmetric in general. e.g. fermion
  - ✓ Variation w.r.t. metric: EMT is symmetric by construction.
  - ✓ Usually, they are related by the improvement term.
* When we solve flow equation $\partial_\lambda \mathscr{L} = \frac{1}{2}\epsilon_{\mu\nu}\epsilon^{\rho\sigma}T^{\mu}{}_{\rho}T^{\nu}{}_{\sigma}$, two EMT give different solutions.
  - ✓ Are they equivalent? (e.g. field redefinition?)
  - ✓ No. doubling of d.o.f. $\mathscr{L} = \frac{i}{2}\psi_+\dot{\psi}_+ + \frac{i}{2}\psi_-\dot{\psi}_- + \cdots + \lambda\psi_+\dot{\psi}_+\psi_-\dot{\psi}_-$

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$T\bar{T}$ Deformation and SUSY

- $T\bar{T}$ deformation $\partial_\lambda \mathscr{L} = \frac{1}{2}\epsilon_{\mu\nu}\epsilon^{\rho\sigma}T^{\mu}{}_{\rho}T^{\nu}{}_{\sigma}$: Not SUSY completed
  - ✓ Does $T\bar{T}$ deformation break SUSY? (perturbative check by [Giraldo-Rivera et al, 2019]
  - ✓ If not, what is the deformed supersymmetry transformation and the deformed supercharge Q?

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$T\bar{T}$ Deformation of Free Scalar Field and Spectrum

Undeformed Free Scalar Field

- Undeformed Lagrangian
  - ✓ $\mathscr{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2$
- From Lagrangian (density) to Hamiltonian (density) by $\Pi = \frac{\delta \mathscr{L}}{\delta \dot{\phi}} = \dot{\phi}$
- Hamiltonian and Momentum
  - ✓ $H_{(0)} = \frac{1}{2}\int dx\, [\Pi^2 + \phi'^2] = \frac{\pi}{L}\sum_k \left[\alpha_{-k}\alpha_k + \bar{\alpha}_{-k}\bar{\alpha}_k\right] = H_+ + H_-$
  - ✓ $P_{(0)} = \int dx\, \Pi\phi' = \frac{\pi}{L}\sum_k \left[\alpha_{-k}\alpha_k - \bar{\alpha}_{-k}\bar{\alpha}_k\right] = H_+ - H_-$

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$T\bar{T}$ Deformation of Free Scalar Field

- Deformed Lagrangian
  - ✓ $$\mathscr{L} = -\frac{1}{2\lambda}\left[\sqrt{1+2\lambda(-\dot{\phi}^2+\phi'^2)}-1\right]$$
- From Lagrangian (density) to Hamiltonian (density) by

$$\Pi = \frac{\delta\mathscr{L}}{\delta\dot{\phi}} = \frac{\dot{\phi}}{\sqrt{1+2\lambda(-\dot{\phi}^2+\phi'^2)}}$$

- Hamiltonian and momentum of Deformed Theory
  - ✓ $$H = \frac{1}{2\lambda}\int_0^L dx\left[\sqrt{1+4\lambda\left(\frac{1}{2}\Pi^2+\frac{1}{2}\phi'^2\right)+4\lambda^2(\Pi\phi')^2}-1\right]$$
  - ✓ $$P = \int dx\,\Pi\phi'$$

Canonical Transformation

* Tempting to conjecture the deformed Hamiltonian (tilde)

✓ $$\widetilde{H} = \frac{L}{2\lambda}\left[\sqrt{1 + \frac{4\lambda}{L}H_{(0)} + \frac{4\lambda^2}{L^2}P_{(0)}^2} - 1\right] = H_+ + H_- + \frac{4\lambda}{L}H_+H_- + \cdots$$

$$H_+ = \frac{\pi}{L}\sum_k \alpha_{-k}\alpha_k$$

$$H_- = \frac{\pi}{L}\sum_k \alpha_{-k}\alpha_k$$

* Compare with the previous Hamiltonian

✓ $$H = \frac{1}{2\lambda}\int_0^L dx\left[\sqrt{1 + 4\lambda\left(\frac{1}{2}\Pi^2 + \frac{1}{2}\phi'^2\right) + 4\lambda^2(\Pi\phi')^2} - 1\right] = H[A, \bar{A}]$$

$$A_k \equiv -\sqrt{\pi}ik\phi_k + \frac{1}{2\sqrt{\pi}}\Pi_k$$

$$\bar{A}_k \equiv -\sqrt{\pi}ik\phi_{-k} + \frac{1}{2\sqrt{\pi}}\Pi_{-k}$$

* For this, we need transformation from A_k , $\bar{A}_k$ to α_k , $\bar{\alpha}_k$ such that $H[A, \bar{A}] = \widetilde{H}[\alpha, \bar{\alpha}]$

  ✓ non-local: one integral vs many integrals

  ✓ canonical: preserve canonical Poisson relations

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Requirements

* The requirement for the map

$$A_k = \alpha_k + \frac{\lambda}{L^2} A_k^{(1)}[\alpha, \bar{\alpha}] + \frac{\lambda^2}{L^4} A_k^{(2)}[\alpha, \bar{\alpha}] + \mathcal{O}(\lambda^3)$$
$$\overline{A}_k = \overline{\alpha}_k + \frac{\lambda}{L^2} \overline{A}_k^{(1)}[\alpha, \bar{\alpha}] + \frac{\lambda^2}{L^4} \overline{A}_k^{(2)}[\alpha, \bar{\alpha}] + \mathcal{O}(\lambda^3)$$

✓ Canonical Transformation

$$[A_k, A_q] = [\overline{A}_k, \overline{A}_q] = \delta_{k+q,0} \ , \qquad [A_k, \overline{A}_q] = 0$$
$$[\alpha_k, \alpha_q] = [\overline{\alpha}_k, \overline{\alpha}_q] = \delta_{k+q,0} \ , \qquad [\alpha_k, \overline{\alpha}_q] = 0$$

They lead to inhomogeneous differential equations for the map.

✓ Hamiltonian and momentum

$$H[A, \bar{A}] = \widetilde{H}[\alpha, \bar{\alpha}]$$
$$P[A, \bar{A}] = \widetilde{P}[\alpha, \bar{\alpha}]$$

From them, one can choose homogeneous solution.

Results

- Solution: $A_k = \alpha_k + \frac{\lambda}{L^2} A_k^{(1)}[\alpha, \bar{\alpha}] + \frac{\lambda^2}{L^4} A_k^{(2)}[\alpha, \bar{\alpha}] + \mathcal{O}(\lambda^3)$

  ✓ $$A_k^{(1)} = 2\pi \sum_{\substack{r,s \\ r+s \neq 0}} \frac{k}{r+s} \alpha_{k-r-s} \bar{\alpha}_{-r} \bar{\alpha}_{-s}$$

  ✓ $$A_k^{(2)} = 2\pi^2 k \sum_{\substack{r,s,u,v \\ u+v \neq 0, r+s \neq 0}} \frac{k-r-s-u-v}{(u+v)(r+s)} \alpha_{k-r-s-u-v} \bar{\alpha}_{-u} \bar{\alpha}_{-v} \bar{\alpha}_{-r} \bar{\alpha}_{-s}$$
  $$-4\pi k \sum_{\substack{u,v \\ u+v \neq 0}} \frac{1}{u+v} \alpha_{k-u-v} \bar{\alpha}_{-u} \bar{\alpha}_{-v} L(H_+ + H_-) + 4\pi^2 k \sum_{\substack{r,s,u,v \\ r+s \neq 0}} \frac{1}{r+s} \alpha_{k-r-s-u-v} \alpha_r \alpha_s \bar{\alpha}_{-u} \bar{\alpha}_{-v}$$

- This is calculated at classical level (up to order $\mathcal{O}(\lambda^2)$).
- At quantum level, it is confirmed up to order $\mathcal{O}(\lambda)$

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$T\bar{T}$ Deformation of Free Fermion and Spectrum

Free Fermion Case

- One can also repeat a similar analysis for the free fermion case.
  - ✓ The final result is similar, but the intermediate procedure is qualitatively different.
  - ✓ worthwhile to present for pedagogical reason.
- Free Fermion
  - ✓ $\mathscr{L}_0 = \frac{i}{2}\psi_+\dot{\psi}_+ + \frac{i}{2}\psi_-\dot{\psi}_- - \frac{i}{2}\psi_+\psi_+' + \frac{i}{2}\psi_-\psi_-'$
- Conjugate momentum?
  - ✓ $\pi_+ = \frac{\overleftarrow{\delta}\mathscr{L}}{\overleftarrow{\delta}\dot{\psi}_+} = \frac{i}{2}\psi_+$ and $\pi_- = \frac{\overleftarrow{\delta}\mathscr{L}}{\overleftarrow{\delta}\dot{\psi}_-} = \frac{i}{2}\psi_-$: There is no $\dot{\psi}_\pm$ on the RHS
  - ✓ forms the second class constraints: $\mathscr{C}_\pm = \pi_\pm - \frac{i}{2}\psi_\pm$

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Dirac Bracket

* Due to the 2nd class constraints $\mathscr{C}_\pm = \pi_\pm - \frac{i}{2}\psi_\pm$, we need to evaluate Dirac bracket.
    * ✓ $\{\mathscr{C}_\pm, \mathscr{C}_\pm\} = -i$ and $\{\mathscr{C}_+, \mathscr{C}_-\} = 0$
    * ✓ For example,

$$\{\psi_+(x_1), \psi_+(x_2)\}_D = 0 - \{\psi_+(x_1), \mathscr{C}_+\}\mathscr{M}^{-1}_{++}\{\mathscr{C}_+, \psi_+(x_2)\} = -i\delta(x_1 - x_2)$$

$T\bar{T}$ Deformation of Free Fermion

- For fermion case, the solution of flow equation is truncated.
  - ✓ Due to the fermi statistics, non-vanishing term is very limited.
    e.g. $\psi_+\partial_{++}\psi_+\psi_-\partial_{=}\psi_-$, $\psi_+\partial_{=}\psi_+\psi_-\partial_{++}\psi_-$
  - ✓ This is because Noether procedure does not produce higher derivative terms
- Deformed Lagrangian
  - ✓ $\mathscr{L} = \frac{i}{2}\psi_+\dot{\psi}_+ + \frac{i}{2}\psi_-\dot{\psi}_- - \frac{i}{2}\psi_+\psi'_+ + \frac{i}{2}\psi_-\psi'_- + \frac{\lambda}{2}\left(-\psi_+\psi'_+\psi_-\dot{\psi}_- + \psi_+\dot{\psi}_+\psi_-\psi'_-\right)$
- Conjugate momentum?
  - ✓ $\pi_+ = \frac{\overleftarrow{\delta}\mathscr{L}}{\overleftarrow{\delta}\dot{\psi}_+} = \frac{i}{2}\psi_+ + \frac{\lambda}{2}\psi_+\psi_-\psi'_-$ and $\pi_- = \frac{\overleftarrow{\delta}\mathscr{L}}{\overleftarrow{\delta}\dot{\psi}_-} = \frac{i}{2}\psi_- - \frac{\lambda}{2}\psi_+\psi'_+\psi_-$: There is no $\dot{\psi}_\pm$, either.
  - ✓ forms the second class constraints:
    $\mathscr{C}_1 = \pi_+ - \frac{i}{2}\psi_+ - \frac{\lambda}{2}\psi_+\psi_-\psi'_-$, $\mathscr{C}_2 = \pi_- - \frac{i}{2}\psi_- + \frac{\lambda}{2}\psi_+\psi'_+\psi_-$

Dirac Bracket

- Dirac bracket of the deformed theory.

$$i\{\psi_+(x_1), \psi_+(x_2)\}_D = (1 + \lambda S_- + 2\lambda^2 S_+ S_-)\delta(x_1 - x_2)$$

$$i\{\psi_+(x_1), \psi_-(x_2)\}_D = -i\lambda(\psi'_+\psi_- + \psi_+\psi'_-)\delta(x_1 - x_2)$$

$$S_\pm = i\psi_\pm\psi'_\pm$$

- Hamiltonian of the deformed theory is
  - ✓ $H = \frac{i}{2}\int dx\left[\psi_+\psi'_+ - \psi_-\psi'_-\right]$: of the same form as free Hamiltonian
  - ✓ no explicit λ dependence

- Then, how does it produce the deformed spectrum?

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Comparison

- Scalar field case
  - ✓ Deformed Hamiltonian has λ dependence
  - ✓ The algebra of phase space variables is not changed.
    : canonical transformation
- Fermion case
  - ✓ Deformed Hamiltonian does not have explicit λ dependence
  - ✓ The algebra of phase space variables is changed.
  - ✓ Not canonical transformation.
  - ✓ This generates λ dependence for the spectrum

Transformation to Free Oscillators

✳ Want: Transformation from $T\bar{T}$ deformed Hamiltonian

$$H = \frac{i}{2}\int dx\left[\psi_+\psi_+' - \psi_-\psi_-'\right]$$

$$i\{\psi_+(x_1), \psi_+(x_2)\}_D = (1 + \lambda S_- + 2\lambda^2 S_+ S_-)\delta(x_1 - x_2)$$
$$i\{\psi_+(x_1), \psi_-(x_2)\}_D = -i\lambda(\psi_+'\psi_- + \psi_+\psi_-')\delta(x_1 - x_2)$$

to Hamiltonian in terms of free oscillators b_k , $\bar{b}_k$

$$\widetilde{H} = \frac{L}{2\lambda}\left[\sqrt{1 + \frac{4\lambda}{L}H_{(0)} + \frac{4\lambda^2}{L^2}P_{(0)}^2} - 1\right] = H_+ + H_- + \frac{4\lambda}{L}H_+H_- + \cdots$$

$$\{b_k, b_q\} = \{\bar{b}_k, \bar{b}_q\} = \delta_{k+q,0}\ , \quad \{b_k, \bar{b}_q\} = 0$$

$$H_+ \equiv -\frac{\pi}{L}\sum_k k b_{-k} b_k$$
$$H_- \equiv \frac{\pi}{L}\sum_k k \bar{b}_{-k}\bar{b}_k$$

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Requirement

* The requirement for the map

$$\psi_k = b_k + \frac{\lambda}{L^2}\psi_k^{(1)}[b,\bar{b}] + \mathcal{O}(\lambda^2) \qquad \bar{\psi}_k = \bar{b}_k + \frac{\lambda}{L^2}\bar{\psi}_k^{(1)}[b,\bar{b}] + \mathcal{O}(\lambda^2)$$

✓ Algebra

$$i\{\psi_+(x_1),\psi_+(x_2)\}_D = (1+\lambda S_- + 2\lambda^2 S_+ S_-)\delta(x_1-x_2)$$
$$i\{\psi_+(x_1),\psi_-(x_2)\}_D = -i\lambda(\psi'_+\psi_- + \psi_+\psi'_-)\delta(x_1-x_2)$$

$$\{b_k, b_q\} = \{\bar{b}_k, \bar{b}_q\} = \delta_{k+q,0} \ , \quad \{b_k, \bar{b}_q\} = 0$$

They lead to inhomogeneous differential equations for the map.

✓ Hamiltonian and momentum

$$H[\psi,\bar{\psi}] = \widetilde{H}[b,\bar{b}] \qquad P[\psi,\bar{\psi}] = \widetilde{P}[b,\bar{b}]$$

From them, one can choose homogeneous solution.

Results

* Solution: $\psi_k = b_k + \frac{\lambda}{L^2}\psi_k^{(1)}[b,\bar{b}] + \mathcal{O}(\lambda^2)$

  ✓ $$\psi_{+,k}^{(1)} = 2\pi \sum_{\substack{r,s \\ r+s\neq 0}} \frac{(k-r-s)s}{r+s} b_{k-r-s} : \bar{b}_r \bar{b}_s : - \pi b_k \sum_r r : \bar{b}_{-r}\bar{b}_r :$$

* This is confirmed at classical level as well as quantum level up to order $\mathcal{O}(\lambda)$.

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Negative Norm States

What happens with Symmetric Energy Momentum Tensor?

- The deformed Lagrangian (by Noether EMT) is
  - ✓ $\mathscr{L} = i\psi_+\partial_=\psi_+ + i\psi_-\partial_{++}\psi_- + \lambda\left(-\psi_+\partial_{++}\psi_+\psi_-\partial_=\psi_- + \psi_+\partial_=\psi_+\psi_-\partial_{++}\psi_-\right)$
  - ✓ Coefficient of quartic term is special in that it does not generate a term $\psi_+\dot{\psi}_+\psi_-\dot{\psi}_-$
- The symmetric EMT by metric variation has different coefficients of the quartic term.
  - ✓ And therefore, this produces the term $\psi_+\dot{\psi}_+\psi_-\dot{\psi}_-$
- What's wrong with this term?

$T\bar{T}$ Deformation of Free Fermion

- From Noether Energy-momentum tensor

$$\mathscr{L} = \frac{i}{2}\psi_+\dot{\psi}_+ + \frac{i}{2}\psi_-\dot{\psi}_- - \frac{i}{2}\psi_+\psi_+' + \frac{i}{2}\psi_-\psi_-' + \frac{\lambda}{2}\left(-\psi_+\psi_+'\psi_-\dot{\psi}_- + \psi_+\dot{\psi}_+\psi_-\psi_-'\right)$$

- From Symmetric Energy-momentum tensor

$$\mathscr{L} = \frac{i}{2}\psi_+\dot{\psi}_+ + \frac{i}{2}\psi_-\dot{\psi}_- - \frac{i}{2}\psi_+\psi_+' + \frac{i}{2}\psi_-\psi_-' + \frac{3\lambda}{8}\left(-\psi_+\psi_+'\psi_-\dot{\psi}_- + \psi_+\dot{\psi}_+\psi_-\psi_-'\right)$$
$$-\frac{\lambda}{8}\psi_+\dot{\psi}_+\psi_-\dot{\psi}_- + \frac{\lambda}{8}\psi_+\psi_+'\psi_-\psi_-'$$

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Emergent D.o.F.

* $T\overline{T}$ deformation of free fermion:

$$\mathscr{L} = \frac{i}{2}\psi_+\dot{\psi}_+ + \frac{i}{2}\psi_-\dot{\psi}_- - \frac{i}{2}\psi_+\psi_+' + \frac{i}{2}\psi_-\psi_-' \boxed{- \frac{\lambda}{8}\psi_+\dot{\psi}_+\psi_-\dot{\psi}_-} + \cdots$$

* Conjugate momentum

$$\pi_+ = \frac{\delta S}{\delta \dot{\psi}_+} = \frac{i}{2}\psi_+ \boxed{- \frac{\lambda}{8}\psi_+\psi_-\dot{\psi}_-}$$

$$\pi_- = \frac{\delta S}{\delta \dot{\psi}_-} = \frac{i}{2}\psi_- \boxed{+ \frac{\lambda}{8}\psi_+\psi_-\dot{\psi}_+}$$

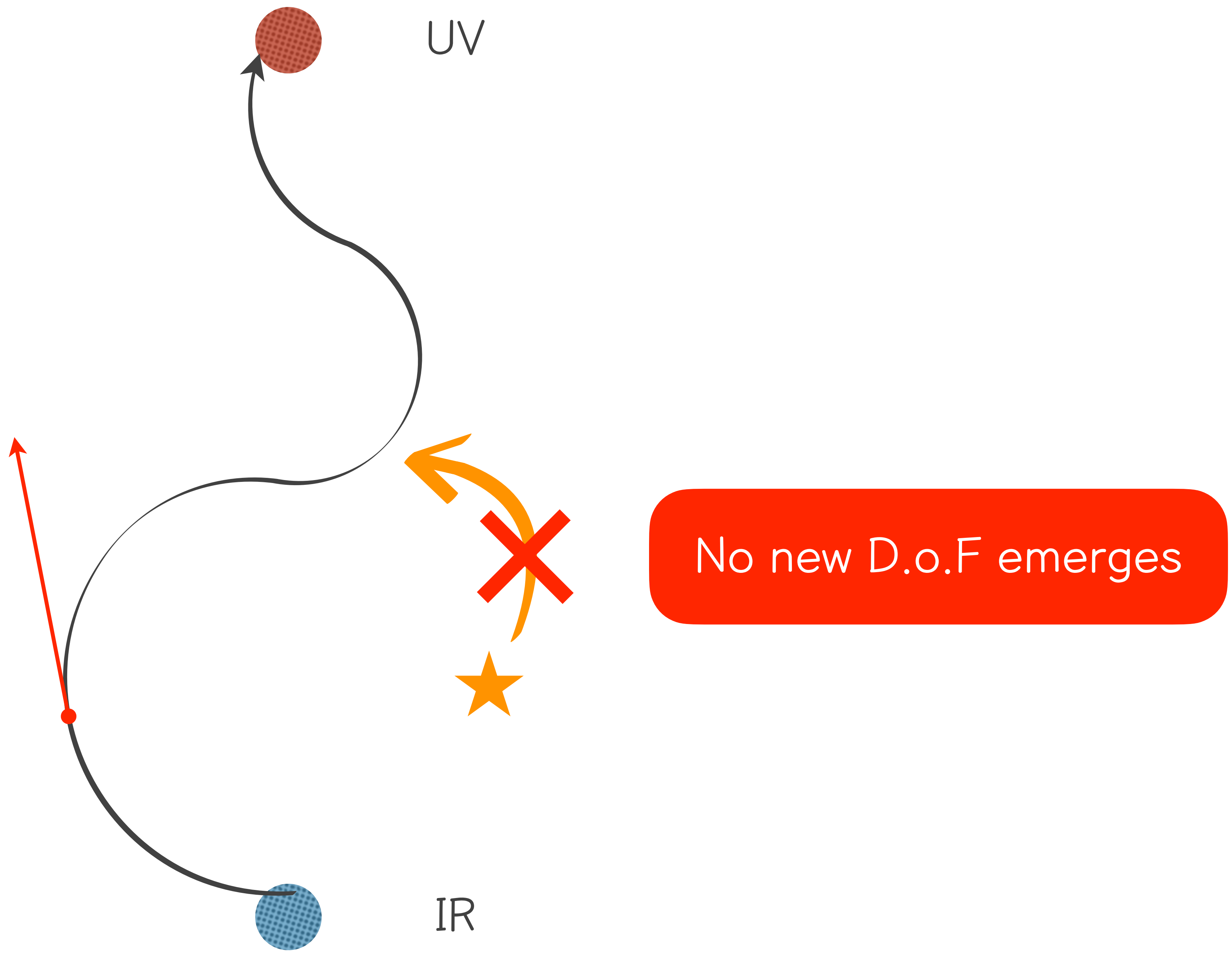
Not constraints any more

✓ RHS contains $\dot{\psi}_-$: Formally, it can be inverted. And it is not constraint any more.

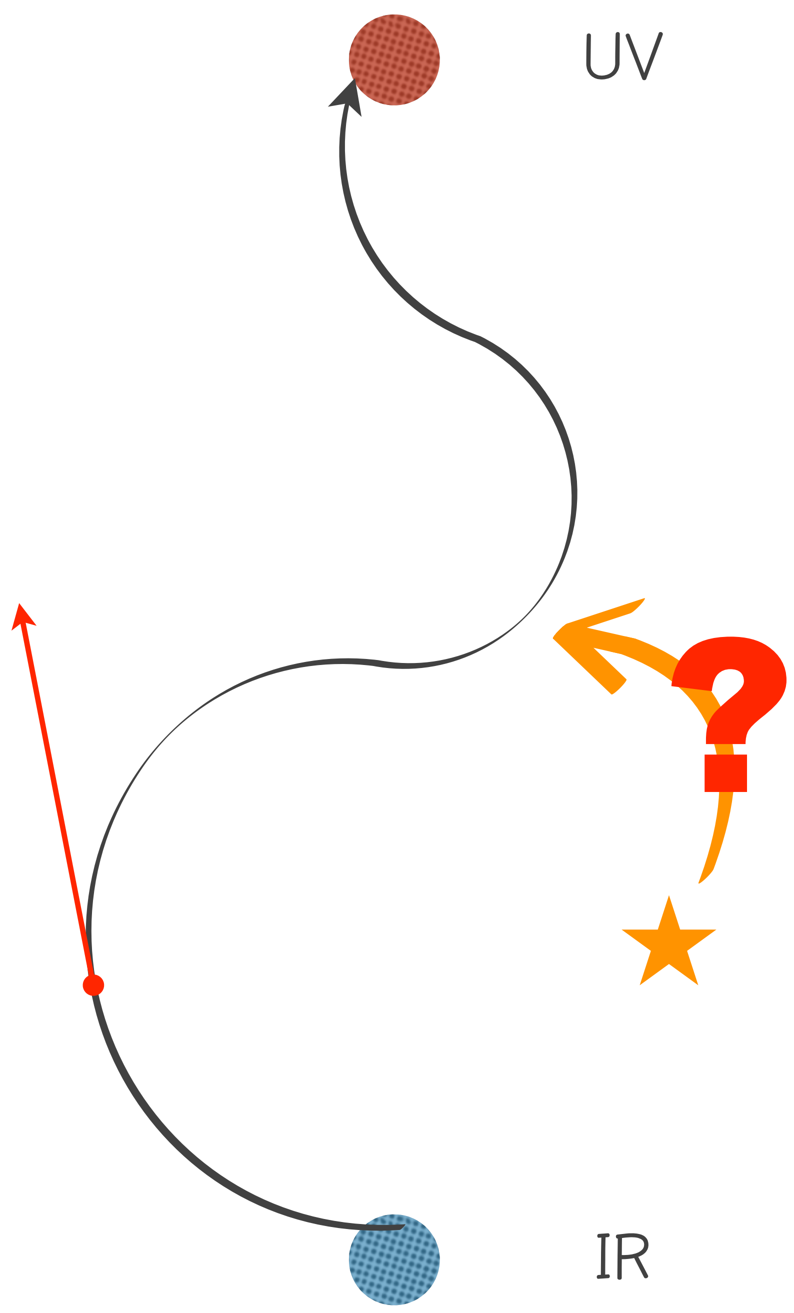
✓ D.o.F, which would have been removed, is now coupled to the system.

✓ Doubling of fermionic D.o.F. : symplectic fermion [A. LeClair and M. Neubert, 2007]
cf) Ostrogradsky instability

Feature 3 of $T\overline{T}$ Deformation 2



Emergent Extra D.o.F.



$$\partial_\lambda \mathscr{L} = \frac{1}{2}\epsilon_{\mu\nu}\epsilon^{\rho\sigma}T^{\mu}{}_{\rho}T^{\nu}{}_{\sigma}$$

$T_{\mu\nu}$ from Noether procedure

"Usually" NO emergent D.o.F

$T_{\mu\nu}$ from Metric variation

"Usually" emergent D.o.F with negative norm

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Toy Model for Negative Norm State

- Quantum mechanical toy model

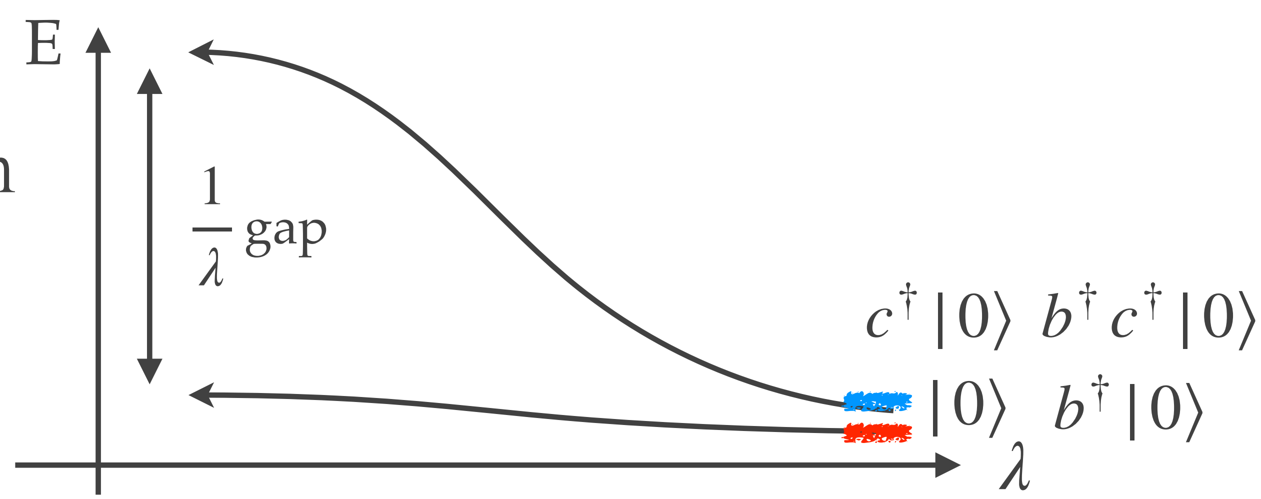
$$L = \frac{i}{2}\bar{\psi}\dot{\psi} - \frac{i}{2}\dot{\bar{\psi}}\psi + m\bar{\psi}\psi - \lambda\dot{\bar{\psi}}\dot{\psi}$$

Hello!

$$\{b, b^\dagger\} = 1$$
$$\{c, c^\dagger\} = -1$$

- Phase space: $\psi\,,\,\bar{\psi}\,,\,\pi\,,\,\bar{\pi}$ → (no constraints) $b\,,\,b^\dagger\,,\,c\,,\,c^\dagger$

- Spectrum



- Negative norm: $\langle 0|cc^\dagger|0\rangle = -1$

  ✓ Non-unitary???

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Recovery of Unitarity

✳ Define J operator: unitary and Hermitian

$$J \equiv 1 + 2c^\dagger c \qquad \begin{aligned} JcJ &= -c \\ Jc^\dagger J &= -c^\dagger \end{aligned} \qquad \begin{aligned} JbJ &= b \\ Jb^\dagger J &= b^\dagger \end{aligned}$$

✳ Define J-inner product

$$\langle \mathcal{O} \rangle_J \equiv \langle J \mathcal{O} \rangle$$

✳ Positive-definite norm: $\langle cc^\dagger \rangle_J = 1$

Deformation of Spectrum

- ✴ Based on the toy model, we expect that the extra D.o.F. would have divergent energy gap in $\lambda \to 0$ limit, and it will be decoupled at $\lambda = 0$.
- ✴ However, the universal formula for the $T\overline{T}$ deformation tells us that there is no divergent energy gap in $\lambda \to 0$ limit.

$$E_n(L,\lambda) = \frac{L}{2\lambda}\left[\sqrt{1+\frac{4\lambda}{L}E_n+\frac{4\lambda^2}{L^2}P_n^2}-1\right]$$

- ✴ Then, what is going on?

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Hermiticity

- ✳ For the case of $T\overline{T}$ deformation, the analysis is very difficult because it is difficult to invert the relation for generic value of λ.

$$\pi_+ = \frac{i}{2}\psi_+ - \lambda\psi_+\psi_-\dot{\psi}_- + \lambda\psi_+\psi_-\psi'_-$$

- ✳ In large λ, one can invert it to express $\dot{\psi}$ in terms of others perturbatively.

New J-inner product → New J-Hermitian

H and P cannot be J-Hermitian in general.

- ✳ The operator J is not uniquely defined.
  - ✓ In quantum mechanical toy model, one can use Bogoliubov transformation to define new J operator where Hamiltonian is J-Hermitian.

Non-Hermiticity

✴ In $T\overline{T}$ deformation of fermion, the operator J is not uniquely defined, and we can make either H or P J-Hermitian by Bogoliubov transformation. But, we cannot make both H and P J-Hermitian at the same time!

H and P cannot be J-Hermitian at the same time!!

➡ $|E, p\rangle$ is not orthogonal

➡ Formula for deformed spectrum is not valid

$$E_n(L,\lambda) = \frac{L}{2\lambda}\left[\sqrt{1+\frac{4\lambda}{L}E_n+\frac{4\lambda^2}{L^2}P_n^2}-1\right]$$

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Choice of Energy-momentum Tensor

- From Noether Energy-momentum tensor

$$\mathscr{L} = \frac{i}{2}\psi_+\dot{\psi}_+ + \frac{i}{2}\psi_-\dot{\psi}_- - \frac{i}{2}\psi_+\psi'_+ + \frac{i}{2}\psi_-\psi'_- + \frac{\lambda}{2}\left(-\psi_+\psi'_+\psi_-\dot{\psi}_- + \psi_+\dot{\psi}_+\psi_-\psi'_-\right)$$

"Good"

- From Symmetric Energy-momentum tensor

$$\mathscr{L} = \frac{i}{2}\psi_+\dot{\psi}_+ + \frac{i}{2}\psi_-\dot{\psi}_- - \frac{i}{2}\psi_+\psi'_+ + \frac{i}{2}\psi_-\psi'_- + \frac{3\lambda}{8}\left(-\psi_+\psi'_+\psi_-\dot{\psi}_- + \psi_+\dot{\psi}_+\psi_-\psi'_-\right)$$

$$-\frac{\lambda}{8}\psi_+\dot{\psi}_+\psi_-\dot{\psi}_- + \frac{\lambda}{8}\psi_+\psi'_+\psi_-\psi'_-$$

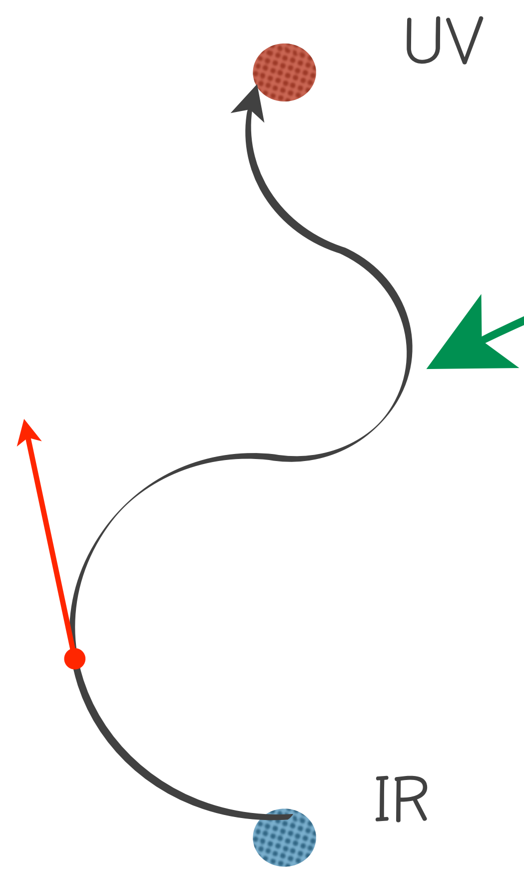
"Bad"

What is the guideline for "good" $T\bar{T}$ Deformation?

String action suggests answer.

$T\bar{T}$ Deformation of Free $\mathcal{N} = (1,1)$ SUSY Model

$T\bar{T}$ Deformation of $\mathcal{N} = (1,1)$ SUSY



Does $T\bar{T}$ deformation preserve $\mathcal{N} = (1,1)$ SUSY?

$$\partial_\lambda \mathscr{L} = \frac{1}{2}\epsilon_{\mu\nu}\epsilon^{\rho\sigma}T^{\mu}{}_{\rho}T^{\nu}{}_{\sigma}$$

$\mathcal{N} = (1,1)$ SUSY

: Deformation operator is not supersymmetric

Bose-Fermi Degeneracy

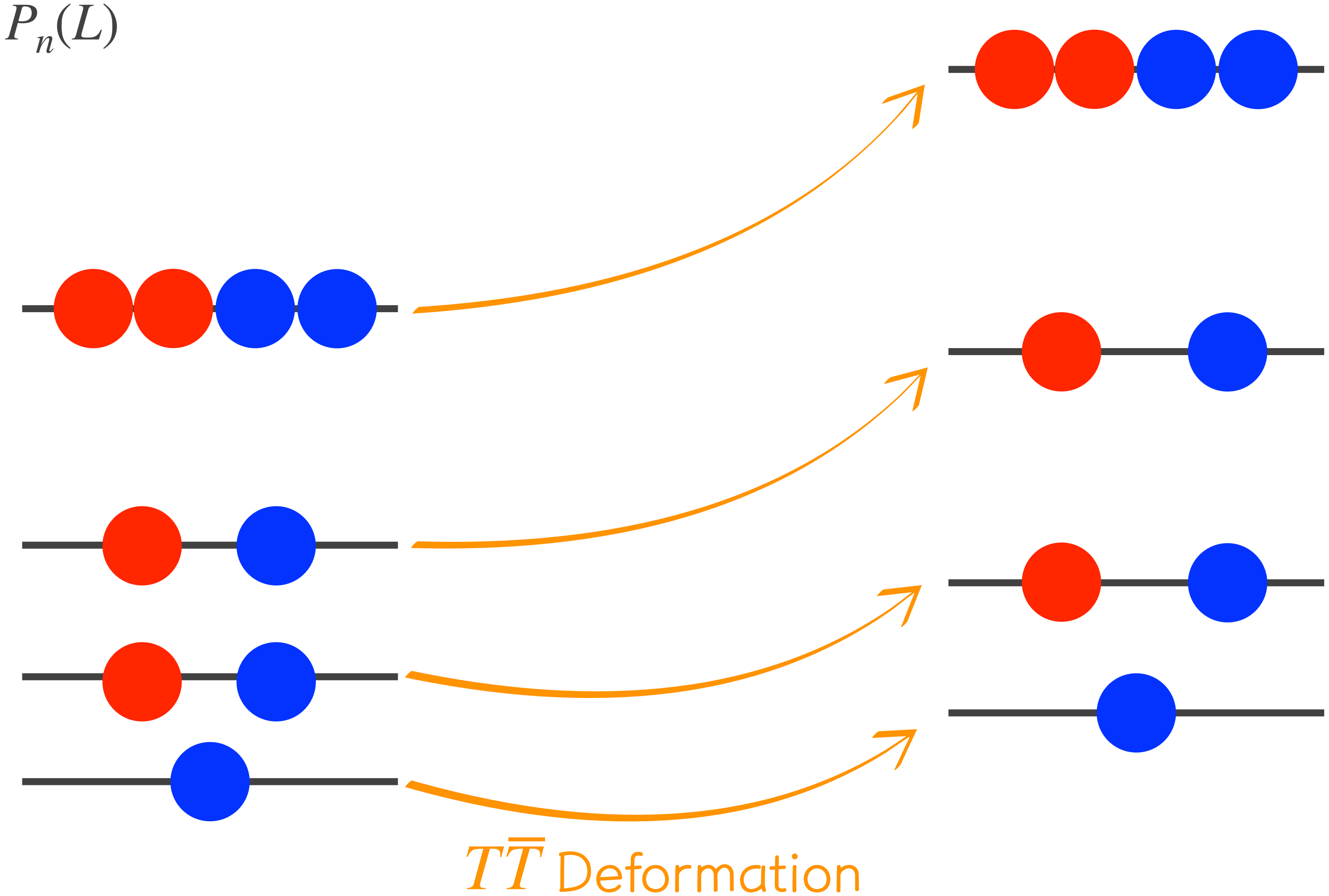
Deformed spectrum both boson and fermion

$$E_n(L,\lambda) = \frac{L}{2\lambda}\left[\sqrt{1+\frac{4\lambda}{L}E_n+\frac{4\lambda^2}{L^2}P_n^2}-1\right]$$

$$P_n(L,\lambda) = P_n(L)$$

E_n : Undeformed Energy

P_n : Undeformed Momentum



Is there local expression for supercharge Q?

What is the deformed supersymmetry transformation?

$$\delta_+\phi = -\frac{1}{2}\psi_+ \quad + \quad ?$$

$T\bar{T}$ Deformation of $\mathcal{N} = (1,1)$ SUSY Model

- Undeformed Lagrangian
  - ✓ $\mathcal{L}_0 = 2\partial_{++}\phi\partial_{=}\phi + i\psi_+\partial_{=}\psi_+ + i\psi_-\partial_{++}\psi_-$
- Solve the flow equation
  - ✓ $\partial_\lambda \mathcal{L} = \frac{1}{2}\epsilon_{\mu\nu}\epsilon^{\rho\sigma}T^{\mu}{}_{\rho}T^{\nu}{}_{\sigma}$ ← $T^{\mu}{}_{\nu}$ from Noether procedure
  - ✓ Solution: $\mathcal{L} = -\frac{1}{2\lambda}\left[\sqrt{1+2\chi}-1\right] + \frac{1+\chi+\sqrt{1+2\chi}}{2\sqrt{1+2\chi}}(S_{++,=} + S_{=,++}) + \cdots$

    $\chi \equiv -4\lambda\partial_{++}\phi\partial_{=}\phi, \quad S_{++,\mu} \equiv i\psi_+\partial_\mu\psi_+, \quad S_{=,\mu} \equiv i\psi_-\partial_\mu\psi_-$
- Conjugate momenta
  - ✓ $\pi = \frac{\delta\mathcal{L}}{\delta\dot{\phi}} \qquad \pi_\pm = \frac{\delta\mathcal{L}}{\delta\dot{\psi}_\pm}$

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Dirac Bracket

- $T\overline{T}$ deformation of $\mathcal{N} = (1,1)$ model

2nd class constraint:

$$\pi_+ - \frac{i}{4}\psi_+\left(1 - 2\lambda\pi\phi' + \sqrt{(1+2\lambda\pi^2)(1+2\lambda\phi'^2)}\right) - \lambda\left[\frac{1+\lambda(\pi^2+\phi'^2)}{4\sqrt{(1+2\lambda\pi^2)(1+2\lambda\phi'^2)}} + \frac{1}{4}\right]\psi_+\psi_-\psi'_- = 0$$

- Dirac brackets

$$\{\phi(x), \pi(y)\}_D = \delta(x-y) \qquad \{\phi(x), \phi(y)\}_D = \{\pi(x), \pi(y)\}_D = 0$$: same

$$i\{\psi_+(x), \psi_+(y)\}_D = \frac{2\lambda\pi\phi' - 1 + \sqrt{(1+2\lambda\pi^2)(1+2\lambda\phi'^2)}}{\lambda(\pi+\phi')^2}\delta(x-y) + \cdots$$

$$i\{\psi_+(x), \psi_-(x)\}_D = -\frac{i\lambda}{\sqrt{(1+2\lambda\pi^2)(1+2\lambda\phi'^2)}}(\psi_+\psi_-)'\delta(x-y)$$

$$\{\phi(x), \psi_+(y)\}_D = \frac{-2\lambda\pi^2 - 1 + \sqrt{(1+2\lambda\pi^2)(1+2\lambda\phi'^2)}}{2(1+2\lambda\pi^2)(\pi+\phi')}\psi_+\delta(x-y)$$

Supercharges

- Supercharges

$$Q_+^1 = \int dx\, \psi_+(\pi + \phi') \qquad Q_-^1 = \int dx\, \psi_-(\pi - \phi')$$

$$\dot{\phi} = \frac{\pi\sqrt{1+2\lambda\phi'^2}}{\sqrt{1+2\lambda\pi^2}} + \cdots \neq \pi$$

$$\{Q_+^1, Q_-^1\}_D = \{Q_\pm^1, H\}_D = \{Q_\pm^1, P\}_D = 0$$

- Hamiltonian and momentum

$$H = \frac{i}{4}\{Q_+^1, Q_+^1\}_D + \frac{i}{4}\{Q_-^1, Q_-^1\}_D$$

$$P = \frac{i}{4}\{Q_+^1, Q_+^1\}_D - \frac{i}{4}\{Q_-^1, Q_-^1\}_D$$

Global Symmetry

- Global symmetry

Shift scalar field $\phi(x) \longrightarrow \phi(x) + a$ ⇒ $\mathbb{P}^2 = \frac{2\pi}{L}\int dx\ \pi$

Shift fermion $\psi_\pm(x) \longrightarrow \psi_\pm(x) + \eta_\pm$ ⇒ $Q^2_\pm = -\frac{8\pi i}{L}\int dx\ \pi_\pm$

$a\,,\ \eta_\pm$: constants

- Commute with Hamiltonian and momentum

$$\{Q^2_\pm, H\}_D = \{\mathbb{P}^2, H\}_D = \{Q^2_\pm, P\}_D = \{\mathbb{P}^2, P\}_D = 0$$

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SUSY and Global Symmetry Algebra

SUSY

$$\{Q^1_\pm, Q^1_\pm\}_D = -2i(H \pm P)$$

$$\{Q^1_+, Q^1_-\}_D = 0$$

Global

$$\{Q^2_\pm, Q^2_\pm\}_D = -\frac{16\pi^2 i}{L} - \frac{16\pi^2 i\lambda}{L^2}(H \mp P)$$

$$\{Q^2_+, Q^2_-\}_D = 0$$

$$\{Q^1_\pm, Q^2_\pm\}_D = -2i\left(\mathbb{P}^2 \pm \frac{4\pi^2}{L^2}\mathbb{W}^2\right) \qquad \mathbb{W}^2 \equiv \frac{L}{2\pi}\oint dx\, \phi'$$

$$\{Q^1_\pm, Q^2_\mp\}_D = 0$$

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Two Approaches for Relation to String Action

- Jorjadze-Theisen approach
  - ✓ Give deformed spectrum ($\widetilde{H}$ in the previous slide) explicitly.
  - ✓ Some issues for SUSY (kappa sym fixing in GS superstring)
  - ✓ λ appears as alpha-prime parameter
- Sfondrini et al and Frolov's approach
  - ✓ Give the deformed Hamiltonian (or equivalently Lagrangian)
  - ✓ We don't know how to derive $\widetilde{H}$.
  - ✓ λ appears as deformation of lightcone coordinates

Green-Schwarz Action

* $\mathcal{N} = 2$ Green-Schwarz Action for 3D target space

$$\mathcal{L}_{GS} = -\frac{1}{2}\gamma^{\alpha\beta}\Pi^{\mu}_{\alpha}\Pi^{\nu}_{\beta}G_{\mu\nu} - i\epsilon^{\alpha\beta}\partial_{\alpha}X^{\mu}(\bar{\Psi}_{+}\Gamma^{\nu}\partial_{\beta}\Psi_{+} - \bar{\Psi}_{-}\Gamma^{\nu}\partial_{\beta}\Psi_{-})G_{\mu\nu} - \epsilon^{\alpha\beta}(\bar{\Psi}_{+}\Gamma^{\mu}\partial_{\alpha}\Psi_{+})(\bar{\Psi}_{-}\Gamma^{\nu}\partial_{\beta}\Psi_{-})G_{\mu\nu}$$

- ✓ $\Pi^{\mu}_{\alpha} \equiv \partial_{\alpha}X^{\mu} + i\bar{\Psi}_{+}\Gamma^{\mu}\partial_{\alpha}\Psi_{+} + i\bar{\Psi}_{-}\Gamma^{\mu}\partial_{\alpha}\Psi_{-}$
- ✓ WZ term: topological
- ✓ spacetime supersymmetry
- ✓ $\Psi_{\pm}$: two component Majorana spinor

"Dictionary"

- Shifted Light-cone coordinates and target space metric

$$X^{+} \equiv \left(\frac{1}{2}-\Lambda\right)X^{1}+\left(\frac{1}{2}+\Lambda\right)X^{0} \qquad X^{-} \equiv X^{1}-X^{0}$$

$$ds^{2} = 2\Lambda(dX^{-})^{2}+2dX^{+}dX^{-}+(dX^{2})^{2}$$

$T\overline{T}$ Deformation parameter Λ

Condition $\Lambda \geqq 0$ is required to demand that X^{+} to be time-like or null

- 3D target coordinates and spinor

Light-cone gauge

$X^{+} = t$: worldsheet time in $T\overline{T}$

X^{-}

$X^{2} = \phi$: scalar field in $T\overline{T}$

$$\Psi_{\pm} = \frac{1}{2}\begin{pmatrix}\psi_{\pm}\\ 0\end{pmatrix}$$

fermion in $T\overline{T}$

gauge fixing of kappa symmetry

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Solving Constraints

- Light-cone gauge : $X^+ = t$

  Charge p_+ for translation of target coordinate X^+ = Hamiltonian of $T\overline{T}$

- Discrete Light-cone quantization : X^- is compactified

  non-trivial topological charge for winding mode

  $$\mathbb{W}^- = \oint d\sigma\, \partial_\sigma X^- = -\,mR$$

  quantized

- Identification of topological charge and momentum in $T\overline{T}$

  $$P = -\frac{2\pi}{L} p_- \mathbb{W}^-$$

  :quantization of operator P in $T\overline{T}$ deformation

  level-matching condition

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SUSY and Global Symmetry Algebra

SUSY

$$\{Q^1_\pm, Q^1_\pm\}_D = -2i(H \pm P)$$

$$\{Q^1_+, Q^1_-\}_D = 0$$

Global

$$\{Q^2_\pm, Q^2_\pm\}_D = -\frac{16\pi^2 i}{L} - \frac{16\pi^2 i\lambda}{L^2}(H \mp P)$$

$$\{Q^2_+, Q^2_-\}_D = 0$$

$$\{Q^1_\pm, Q^2_\pm\}_D = -2i\left(\mathbb{P}^2 \pm \frac{4\pi^2}{L^2}\mathbb{W}^2\right) \qquad \mathbb{W}^2 \equiv \frac{L}{2\pi}\oint dx\,\phi'$$

$$\{Q^1_\pm, Q^2_\mp\}_D = 0$$

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3D Target Space SUSY

- $\mathcal{N} = 2$ SUSY of 3D target space

$Q^1_\pm$: $\mathcal{N} = (1,1)$ supercharge

$Q^2_\pm$: fermionic global charge

$$\{Q^\alpha_a, Q^\beta_b\}_D = -2i\delta_{ab}(\Gamma^\mu C)^{\alpha\beta}\mathbb{P}_\mu - \frac{2i}{2\pi\ell_s^2}\sigma^3_{ab}A^{\alpha\beta}$$

topological charge from WZ term

Hamiltonian

$$\Gamma^\mu C\mathbb{P}_\mu = \begin{pmatrix} H & \mathbb{P}^2 \\ \mathbb{P}^2 & \frac{8\pi^2}{L} + 2\Lambda H \end{pmatrix} \qquad A = \Gamma_\mu C \oint d\sigma\, \partial_\sigma X^\mu = \frac{L^2}{4\pi^2}\begin{pmatrix} P & \frac{4\pi^2}{L^2}\mathbb{W}^2 \\ \frac{4\pi^2}{L^2}\mathbb{W}^2 & -2\Lambda P \end{pmatrix}$$

3D Target Space SUSY

- $\mathcal{N} = 2$ SUSY of 3D target space

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$$\{Q^\alpha_a, Q^\beta_b\}_D = -2i\delta_{ab}(\Gamma^\mu C)^{\alpha\beta}\mathbb{P}_\mu - \frac{2i}{2\pi\ell_s^2}\sigma^3_{ab}A^{\alpha\beta}$$

topological charge from WZ term

$$\Gamma^\mu C \mathbb{P}_\mu = \begin{pmatrix} H & \mathbb{P}^2 \\ \mathbb{P}^2 & \frac{8\pi^2}{L} + 2\Lambda H \end{pmatrix} \qquad A = \Gamma_\mu C \oint d\sigma\, \partial_\sigma X^\mu = \frac{L^2}{4\pi^2}\begin{pmatrix} P & \frac{4\pi^2}{L^2}\mathbb{W}^2 \\ \frac{4\pi^2}{L^2}\mathbb{W}^2 & -2\Lambda P \end{pmatrix}$$

$\mathbb{P}^2$: bosonic global charge

3D Target Space SUSY

- $\mathcal{N} = 2$ SUSY of 3D target space

$Q^1_\pm$: $\mathcal{N} = (1,1)$ supercharge

$Q^2_\pm$: fermionic global charge

$$\{Q^\alpha_a, Q^\beta_b\}_D = -2i\delta_{ab}(\Gamma^\mu C)^{\alpha\beta}\mathbb{P}_\mu - \frac{2i}{2\pi\ell_s^2}\sigma^3_{ab}A^{\alpha\beta}$$

topological charge from WZ term

$$\Gamma^\mu C\mathbb{P}_\mu = \begin{pmatrix} H & \mathbb{P}^2 \\ \mathbb{P}^2 & \frac{8\pi^2}{L} + 2\Lambda H \end{pmatrix} \qquad A = \Gamma_\mu C\oint d\sigma\,\partial_\sigma X^\mu = \frac{L^2}{4\pi^2}\begin{pmatrix} P & \frac{4\pi^2}{L^2}\mathbb{W}^2 \\ \frac{4\pi^2}{L^2}\mathbb{W}^2 & -2\Lambda P \end{pmatrix}$$

P : momentum

3D Target Space SUSY

- $\mathcal{N} = 2$ SUSY of 3D target space

$Q^1_\pm$: $\mathcal{N} = (1,1)$ supercharge

$Q^2_\pm$: fermionic global charge

$$\{Q^\alpha_a, Q^\beta_b\}_D = -2i\delta_{ab}(\Gamma^\mu C)^{\alpha\beta}\mathbb{P}_\mu - \frac{2i}{2\pi\ell_s^2}\sigma^3_{ab}A^{\alpha\beta}$$

topological charge from WZ term

$$\Gamma^\mu C\mathbb{P}_\mu = \begin{pmatrix} H & \mathbb{P}^2 \\ \mathbb{P}^2 & \frac{8\pi^2}{L} + 2\Lambda H \end{pmatrix} \qquad A = \Gamma_\mu C \oint d\sigma\, \partial_\sigma X^\mu = \frac{L^2}{4\pi^2}\begin{pmatrix} P & \frac{4\pi^2}{L^2}\mathbb{W}^2 \\ \frac{4\pi^2}{L^2}\mathbb{W}^2 & -2\Lambda P \end{pmatrix}$$

: topological charge for compactified boson

Comments

* Partially broken rigid SUSY

$$\{Q^2_\pm, Q^2_\pm\}_D = -\frac{16\pi^2 i}{L} - \frac{16\pi^2 i\lambda}{L^2}(H \mp P)$$

  - ✓ Due to topological charge
  - ✓ fermionic global symmetry in $T\overline{T}$ deformation

* BPS States

  - ✓ BPS states from the point of view of 3D $\mathcal{N} = 2$ SUSY
  - ✓ Protected along $T\overline{T}$ deformation

$$E = \frac{L}{2\lambda}\left[\sqrt{1 + \frac{4\lambda}{L}\left|P - \frac{1}{L}p_m p_w\right| + \frac{2\lambda}{L^2}\left(\frac{L^2}{4\pi^2}p_m^2 + \frac{4\pi^2}{L^2}p_w^2\right) + \frac{4\lambda^2}{L^2}P^2} - 1\right]$$

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Future Works

- ✳ More systematic way to find the map perturbatively.
  cf [Theisen, Jorjadze, 2020]
  - ✓ It will be useful to evaluate the deformation of correlation functions.
- ✳ It is tempting to relate extra negative norm state with the other branch of the deformation of spectrum which is divergent in $\lambda \to 0$ limit.

$$E_n(L,\lambda) = \frac{L}{2\lambda}\left[-\sqrt{1+\frac{4\lambda}{L}E_n+\frac{4\lambda^2}{L^2}P_n^2}-1\right]$$

  - ✓ But, it might be "wishful" thinking.
- ✳ Further investigation on the relation to string actions.
  - ✓ Other winding sectors, other backgrounds

Thank You