# Elliptic functions and exact summation of leading logs around beyond $T\bar{T}$ deformation of $O(N+1)$-symmetric 2D QFTs

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НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ САНКТ-ПЕТЕРБУРГ

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НИЦ КИ

# Outline

1. Introduction;
2. Recurrence relations for leading logarithms (LLs) from unitarity, analyticity and crossing symmetry in $D = 2$;
3. Pseudofactorials and quasi-renormalizable QFTs;
4. LLs around a beyond $T\bar{T}$ deformation of the O($N$)-symmetric free theory in $D = 2$;
5. LLs around a beyond $T\bar{T}$ deformation of the O($N + 1$)/O($N$) $\sigma$-model in $D = 2$;
6. Summary and Outlook.

Based on

M. Polyakov, A. Smirnov, K.S. and A. Vladimirov, Theor. Math. Phys. **200** (2019) 1176 [arXiv:1811.08449 [hep-th]];
J. Linzen, M. Polyakov, K.S. and N. Sokolova, JHEP **1904** (2019) 007, [arXiv:1811.12289 [hep-ph]];
J. Linzen, M. Polyakov, K.S. and N. Sokolova, JHEP **05** (2021), 266 [arXiv:2104.01038 [hep-th]].

# Maxim V. Polyakov (1966-2021)

# LLogs in $\chi$PT ($SU(2)_V \times SU(2)_A \to SU(2)_I$)

$$\mathcal{L}_{\chi\mathrm{PT}} = \mathcal{L}_2 + \mathcal{L}_4 + \ldots .$$

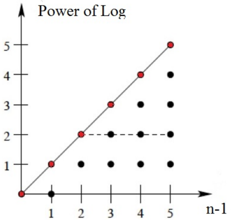
- Infinite component EFTs are renormalizable: S.Weinberg'79.
- $2 \to 2$ scattering amplitude:

$$A(s,t) = \underbrace{c_1 \frac{E^2}{\Lambda^2}}_{\text{only } \mathcal{L}_2 \text{ parameters}} + \frac{E^4}{\Lambda^4}\Bigg( \underbrace{c_2 \log\left(\frac{\mu^2}{E^2}\right)}_{\substack{1-\text{loop} \\ \text{only} \\ \mathcal{L}_2 \text{ parameters}}} + \underbrace{c_3}_{\substack{1-\text{loop} \quad \mathcal{L}_2 \quad \text{parameters} \\ \text{tree}-\text{level} \quad \mathcal{L}_4 \quad \text{parameters}}} \Bigg) + O\left(\frac{E^6}{\Lambda^6}\right);$$

## Leading Logs

$n \equiv$ [number of loops $+1$]

$$A(s,t) = \sum_{n=1}^{\infty} \omega_n \left(\frac{E^2}{\Lambda^2}\right)^n \log^{n-1}\left(\frac{\mu^2}{E^2}\right) + \mathrm{NLL}$$



# Do we really have to care about LLogs in EFTs?

## Renormalizable QFT

$$A(s) = \alpha + \alpha^2(a_1 \log|s| + b_1) + \alpha^3(a_2 \log^2|s| + b_2 \log|s| + c) + \ldots; \quad \alpha \sim \frac{\alpha_0}{\log|s|};$$

- LLog approximation gives leading asymptotic behavior.

## Case of $\chi$PT

$$A(s) = \frac{s}{F^2} + \frac{s^2}{F^4}(a_1 \log|s| + b_1) + \frac{s^3}{F^6}(a_2 \log^2|s| + b_2 \log|s| + c) + \ldots;$$
$$s \ll F^2; \quad s \log|s| \sim s.$$

- In $\chi$PT LLog can not compete with power-like corrections. No reason to be particularly interested in their resummation... But!

- Some phenomenology in $D = 4$: I. Perevalova, M. Polyakov, A. Vall and A. Vladimirov'11;
- LLs can mirror some fundamental properties of the theory!

# LLs for EFTs in $D = 4$

## Some references:

- D. Kazakov'88 and M. Buchler, G. Colangelo'04: generalization of the RG-group methods for EFTs.
- M. Bissegger, A. Fuhrer'07: 5-loop LLs in massless O(4)/O(3) $\sigma$-model;
- N. Kivel, M. Polyakov, A. Vladimirov'08.

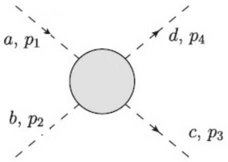
- J. Koschinski , M. Polyakov, A. Vladimirov'10: arbitrary-loop LLs in massless $\Phi^4$-type EFTs from unitarity, analyticity and crossing;
- Action ($\Phi^4$-type theory; all fields are massless *e.g.* $O(N+1)/O(N)$ $\sigma$-model):

$$S = \int d^D x \left[ \frac{1}{2} \partial_\mu \Phi^a \partial^\mu \Phi^a - V(\Phi, \partial\Phi) \right];$$

- The lowest chiral order of interaction is $2\kappa$: *e.g.* $\Phi^2 \partial^{2\kappa} \Phi^2$;
- The Lagrangian is invariant under some particular global symmetry group $G$ (isospin);
- Infinite system of recurrence relations for LL-coefficients of PWs: generalization of the RG-equations;
- Too hard to solve analytically in $D = 4$, but much simpler in $D = 2$.

## Main object of study: $2 \to 2$ scattering amplitude

- Symmetry group $O(N)$;
- $2 \to 2$ scattering amplitude:




$$\langle \Phi_c(p_3)\,\Phi_d(p_4)\,|S-1|\Phi_a(p_1)\,\Phi_b(p_2)\rangle = i(2\pi)^2 \frac{s}{2\sqrt{s(s-4m^2)}}$$
$$\times \left[\delta(\mathbf{p}_1-\mathbf{p}_4)\,\delta(\mathbf{p}_2-\mathbf{p}_3)\sum_I P^I_{abcd}\mathcal{M}^{I,T}(s) + \delta(\mathbf{p}_1-\mathbf{p}_3)\,\delta(\mathbf{p}_2-\mathbf{p}_4)\sum_I P^I_{abcd}\mathcal{M}^{I,R}(s)\right].$$

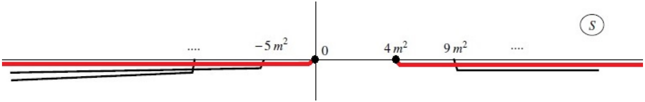
- $\mathcal{M}^{I,\{T,R\}}(s)$: transition and reflection amplitudes;
- $P^I_{abcd}$ are projectors on the invariant isospin subspaces:

$$P^{I=0}_{abcd} = \frac{\delta_{ab}\delta_{cd}}{N};\quad P^{I=1}_{abcd} = \frac{\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc}}{2};\quad P^{I=2}_{abcd} = \frac{\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}}{2} - \frac{1}{N}\delta_{ab}\delta_{cd};$$

# LLs in massless EFT in $D = 2$: preliminaries

- LLog structure of PW amplitudes:

$$\mathcal{M}^{I,\{T,R\}}(s) = 4\pi s \sum_{n=1}^{\infty} \hat{S}^n \sum_{i=0}^{n-1} \alpha_{n,i}^{I,\{T,R\}} \log^i\left(\frac{\mu^2}{s}\right) \log^{n-i-1}\left(\frac{\mu^2}{-s}\right) + \mathcal{O}(\mathrm{NLL});$$




- $F$ is the coupling in the Lagrangian; $[F] = \kappa - 1$
- $\hat{S} = \frac{s^{\kappa-1}}{4\pi F^2}$ is the dimensionless expansion parameter.
- Both left and right cuts in the complex $s$-plane contribute.
- The LL coefficients are given by

$$\omega_n^{I,\{T,R\}} = \sum_{i=0}^{n-1} \alpha_{n,i}^{I,\{T,R\}};$$

# Recurrence relation for $\omega_n^{I,\{T,R\}}$ from unitarity, analyticity and crossing I

- Only 2-particle unitarity is relevant! 2-particle unitarity ($s$-channel cut):

$$\mathrm{Im}\,\mathcal{M}^T(s)\big|_{s>0} = \frac{1}{8s}\frac{s}{\sqrt{s(s-4m^2)}}\left(\left|\mathcal{M}^T(s)\right|^2 + \left|\mathcal{M}^R(s)\right|^2\right)$$
$$\mathrm{Im}\,\mathcal{M}^R(s)\big|_{s>0} = \frac{1}{8s}\frac{s}{\sqrt{s(s-4m^2)}}\left(\mathcal{M}^{T^*}(s)\mathcal{M}^R(s) + \mathcal{M}^{R^*}(s)\mathcal{M}^T(s)\right)$$

- Implication of the $s$-channel cut unitarity:

$$\sum_{i=0}^{n-1} \alpha_i^{I,T}(n-i-1) = \frac{1}{2}\sum_{k=1}^{n-1}\left(\omega_k^{I,T}\omega_{n-k}^{I,T} + \omega_k^{I,R}\omega_{n-k}^{I,R}\right);$$

$$\sum_{i=0}^{n-1} \alpha_i^{I,R}(n-i-1) = \frac{1}{2}\sum_{k=1}^{n-1}\left(\omega_k^{I,T}\omega_{n-k}^{I,R} + \omega_k^{I,R}\omega_{n-k}^{I,T}\right);$$

# Recurrence relation for $\omega_n^{I,\{T,R\}}$ from unitarity, analyticity and crossing II

- Crossing symmetry and the isospin crossing matrices ($s \leftrightarrow u$-crossing):

$$\mathcal{M}^I(s,t,u) = C_{su}^{IJ} \mathcal{M}^J(u,t,s); \quad \mathcal{M}^I(s,t,u) = C_{st}^{IJ} \mathcal{M}^J(t,s,u);$$

- Crossing matrices expressed through the projectors

$$C_{su}^{IJ} = P_{abcd}^I P_{bdac}^J \frac{1}{d_I}; \quad C_{st}^{IJ} = \frac{1}{d_I} P_{abcd}^I P_{cbad}^J;$$

- Analyticity + crossing :

$$\mathcal{M}^I\left(s, t=0, u=4m^2-s\right) \equiv \mathcal{M}^{I,T}(s) = \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \sum_J \left( \frac{\delta^{IJ}}{s'-s} + \frac{C_{su}^{IJ}}{s'-u} \right) \operatorname{disc} \mathcal{M}^{J,T}\left(s'\right)$$

$$\mathcal{M}^I\left(s, t=4m^2-s, u=0\right) \equiv \mathcal{M}^{I,R}(s) = \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \sum_J \left( \frac{\delta^{IJ}}{s'-s} + \frac{C_{st}^{IJ}}{s'-t} \right) \operatorname{disc} \mathcal{M}^{J,T}\left(s'\right)$$

# Recurrence relation for $\omega_n^{I,\{T,R\}}$ from unitarity, analyticity and crossing III

- 2-particle unitarity + analyticity (Roy equations):

$$\mathrm{Im}\mathcal{M}^{I,\,T}(s)\Big|_{s<0} = \sum_J C_{su}^{IJ} \frac{1}{8s} \frac{s}{\sqrt{-s(-s-4m^2)}} \left( |\mathcal{M}^{J,\,T}(-s)|^2 + |\mathcal{M}^{J,\,R}(-s)|^2 \right);$$

$$\mathrm{Im}\mathcal{M}^{I,\,R}(s)\Big|_{s<0} = \sum_J C_{st}^{IJ} \frac{1}{8s} \frac{s}{\sqrt{-s(-s-4m^2)}} \left( \mathcal{M}^{J,\,T*}(-s)\mathcal{M}^{J,\,R}(-s) + \mathcal{M}^{J,\,R*}(-s)\mathcal{M}^{J,\,T}(-s) \right).$$

- Closed form of the recurrence relations

$$\omega_n^{I,\,T} = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \sum_J \left( \delta^{IJ} - (-1)^{(\kappa-1)n} C_{su}^{IJ} \right) \left( \omega_k^{J,\,T} \omega_{n-k}^{J,\,T} + \omega_k^{J,\,R} \omega_{n-k}^{J,\,R} \right);$$

$$\omega_n^{I,\,R} = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \sum_J \left( \delta^{IJ} - (-1)^{(\kappa-1)n} C_{st}^{IJ} \right) \left( \omega_k^{J,\,T} \omega_{n-k}^{J,\,R} + \omega_k^{J,\,R} \omega_{n-k}^{J,\,T} \right).$$

- Reminder for the indices:

$n$ : number of loops $+1$;  $I,\,J$ : label isospin invariant subspaces;

- Initial conditions $\omega_{n=1}^{I,\{T,R\}}$ come from the tree-level calculation.

## Case of renormalizable theory: $O(N)$-symmetric $\Phi^4$ in $D=4$

- Consider

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^i \partial^\mu \Phi^i - \frac{\lambda_0}{4!}(\Phi^2)^2.$$

- LLog approximation can be obtained by solving 1-loop RG equation:

$$\mu^2 \frac{\partial}{\partial \mu^2}\lambda(\mu^2) = \beta(\lambda) = \frac{N+2}{8}\lambda^2(\mu^2) + \mathcal{O}(\lambda^2) \Rightarrow A(s,t) = \lambda(\mu^2) = \frac{\lambda_0}{1 - b_1 \lambda_0 \log(\mu^2/s)}$$

- Same solution comes from the recursive equations.
- The crossing matrices form the 1-loop $\beta$-function coefficient. **N.b.** $\kappa = 0$.

$$\omega_n = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \underbrace{\frac{N+2}{8}}_{b_1} \omega_k \omega_{n-k} \qquad \Rightarrow A(s,t) = \sum_{n=1}^{\infty} \lambda_0^n \log^{n-1}\left(\frac{\mu^2}{s}\right)\omega_n$$

## Simplified form of the recurrence relation

**Formulation of the problem:**

- Generating function:

$$f(z) = \sum_{n=1}^{\infty} f_n z^{n-1};$$

- Non-linear recurrence relation:

$$f_n = \frac{1}{n-1} \sum_{k=1}^{n-1} A(n,k) f_k f_{n-k}, \quad f_1 = 1.$$

- $A(n,k)$ function: Greek "$\mathcal{A}\nu\alpha\delta\rho o\mu\acute{\eta}$" for "recursion" (courtesy of N. Stefanis).
- Singularities of $f(z)$ closest to the origin play the crucial role for the asymptotic behavior of $f_n$ for $n \to \infty$.

# Some remarkable cases I: Catalan numbers

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}, \quad C_0 = 1, \quad A(n+1,k) = n; \quad f(z) = \sum_{n=0}^{\infty} C_n z^n = \frac{2}{1+\sqrt{1-4z}}.$$

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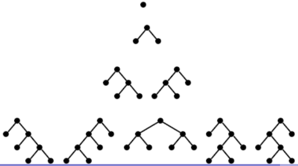
Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A000108 Catalan numbers: C(n) = binomial(2n,n)/(n+1) = (2n)!/(n!(n+1)!). Also called Segner numbers. 2656
(Formerly M1459 N0577)

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, 18367353072152, 69533550916004,

- Plenty of combinatoric applications.
- *E.g.* number of rooted binary trees with $n$ internal nodes and $n+1$ external nodes



## Some remarkable cases II: Bessel functions

A. Vladimirov, 2010:

$$\boxed{A(n,k)=\frac{n-1}{n+\nu};}\quad \nu-\text{ parameter.}$$

- The solution:

$$f(z)=\sqrt{\frac{\nu+1}{z}}\frac{J_{\nu+1}(2\sqrt{z(\nu+1)})}{J_{\nu}(2\sqrt{z(\nu+1)})}.$$

- For $\nu>-1$ poles along $z>0$ axis and a cut along $z<0$;
- For $\nu<-1$ poles along $z<0$ axis and a cut along $z>0$;
- N.b. $\nu=\frac{1}{2}$: $f(z)=\frac{1-\sqrt{6z}\cot\sqrt{6z}}{2z}$ *c.f.* D. Kazakov: summing up UV-divergencies in the supersymmetric gauge theories ($D=6,\,8,\,10$ SYM).

### Motivation from A.A. Migdal

- A.A.'Migdal'1977, 78: 2-point function of large-$N_c$ QCD as the sum of infinite number of pole terms with spectrum given by roots of the Bessel functions;
- $AdS/CFT$: same spectrum reported (see J.Erlich et al.'2006).

# More remarkable examples: factorials and pseudofactorials

Case of renormalizable theory

- $A(n,k) = 1$: factorial:
  $\{a_n\} \equiv \{f_n (n-1)!\} = \{0!, 1!, 2!, \ldots\}$

$$f'(z) = -f^2(z); \quad f(0) = 1.$$

Quasi-renormalizable theory ??

- $A(n,k) = (-1)^{n+1}$
  R. Bacher and P. Flajolet'09: pseudofactorial sequence:
  $\{a_n\} \equiv \{f_n (n-1)!\} = \{1, -1, -2, 2, 16, -40, -320, 1040, \ldots\}$.

$$f'(z) = -f^2(-z); \quad f(0) = 1.$$

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Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

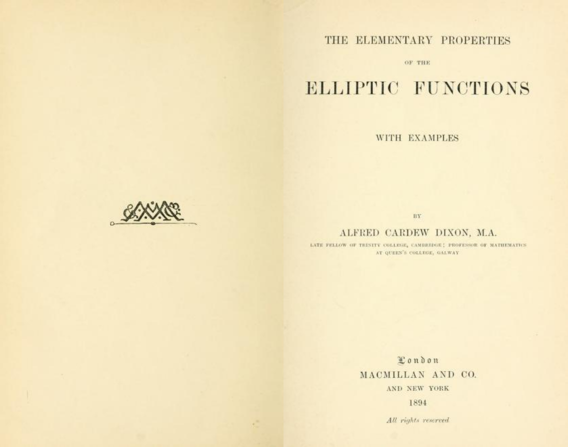
A098777 Pseudo-factorials: a(0)=1, a(n+1) = (-1)^(n+1) * Sum_{k=0..n} binomial(n,k) * a(k)*a(n-k), n>=0. 4

1, -1, -2, 2, 16, -40, -320, 1040, 12160, -52480, -742400, 3872000, 66457600, -411136000, -8202444800, 58479872000, 1335009280000, -10791497728000, -277035646976000, 2502527565824000, 7139193487360000, -712816377856000000, -22367684235100160000, 24459723607801856000 (list; graph; refs; listen; history; text; internal format)

# Dixon's elliptic functions

Dixon's elliptic functions (Dixon'189X) (elliptic = meromorphic, doubly periodic)

$$\begin{cases} \mathrm{sm}'(z) = \mathrm{cm}^2(z) \\ \mathrm{cm}'(z) = -\mathrm{sm}^2(z) \end{cases} \qquad \mathrm{sm}(0) = 0; \quad \mathrm{cm}(0) = 1 \qquad \boxed{\mathrm{sm}^3(z) + \mathrm{cm}^3(z) = 1.}$$


THE ELEMENTARY PROPERTIES

OF THE

ELLIPTIC FUNCTIONS

WITH EXAMPLES

BY

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MACMILLAN AND CO.

AND NEW YORK

1894

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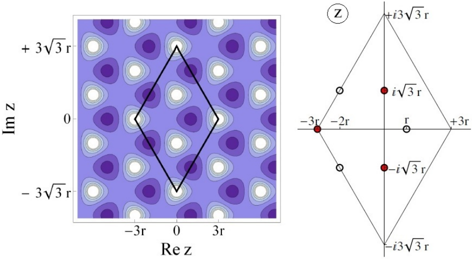
# Dixon's elliptic functions and pseudofactorial

The equation is equivalent to $(g(z) \equiv f(-z))$

$$\begin{cases} f'(z) = -g^2(z) \\ g'(z) = f^2(z), \end{cases} \qquad f(0) = 1; \quad g(0) = 1.$$

- This system has a simple first integral: $f^3(z) + g^3(z) = 2$.

Integration gives: $$f(z) = 2^{\frac{1}{3}} \mathrm{sm}(\frac{\pi_3}{6} - 2^{\frac{1}{3}} z),$$ with $$\frac{\pi_3}{6} = \frac{1}{6} B(\frac{1}{3}, \frac{1}{3}).$$




- Real period: $\omega_3 \equiv 6r = 2^{-1/3}\pi_3$. Invariance under rotations by $\pm\frac{2\pi}{3}$.
- Can be expressed through the Weierstrass $\wp$-function.

# Landau pole in QFT and quasi-renormalizable QFTs

- Running coupling through the renormalized coupling $\lambda_R \equiv \lambda(\mu_0)$:

$$\lambda(\mu) = \frac{\lambda_R}{1 - b_1 \lambda_R \log \frac{\mu}{\mu_0}}.$$

- $\beta$-function

$$\beta(\lambda) = \lambda\Big( \underbrace{\frac{3}{16\pi^2}}_{b_1} \lambda + \dots \Big).$$

- Dimensional transmutation

$$E_\infty = \mu_0 \exp\left( \int_{\lambda_R}^{\infty} \frac{d\lambda}{\beta(\lambda)} \right); \quad E_\infty \Big|_{\frac{\lambda \Phi^4}{4!}} = \mu_0 \exp\left( \frac{16\pi^2}{3\lambda_R} \right);$$

**Quasi-renormalizable QFT**

We call the quantum field theory **quasi-renormalizable** if the generating function for the coefficients of leading logs of $2 \to 2$ scattering amplitude (defined from the recurrence relation) is a meromorphic function of the variable $z \equiv \log\left(\frac{\mu^2}{s}\right)$.

## Quasi-renormalizable QFTs. O($N$)-symmetric bi-quartic theory in $D = 2$

- O($N$)-symmetric bi-quartic ($\Phi^4$-type with 4 derivatives) theory:

$$S = \int d^2x \Big[ \frac{1}{2} \partial_\mu \Phi^a \partial^\mu \Phi^a - g_1 (\partial_\mu \Phi^a \partial^\mu \Phi^a)(\partial_\nu \Phi^b \partial^\nu \Phi^b) - g_2 (\partial_\mu \Phi^a \partial_\nu \Phi^a)(\partial^\mu \Phi^b \partial^\nu \Phi^b) \Big].$$

- LL-approximation of the O($N$)-symmetric bi-quartic theory is infrared finite in $D = 2$.
- Can be seen as a general deformation of the free O($N$)-symmetric theory with a generic dimension-4 operator:

$$S = \int d^2x \Big[ \frac{1}{2} \partial_\mu \Phi^a \partial^\mu \Phi^a - 4\lambda \det T_{\mu\nu} - \frac{1}{G} (\partial_\mu \Phi^a \partial_\nu \Phi^a)(\partial^\mu \Phi^b \partial^\nu \Phi^b) \Big].$$

$$\frac{1}{G} = 2g_1 + g_2; \quad \lambda = -g_1; \quad [G] = 2; \quad [\lambda] = -2.$$

N.B. $2g_1 + g_2 = 0$: pure $T\bar{T}$ deformation - no Logs!

## LLog coefficients in $D = 2$ bi-quartic theory I

- Explicit form of recurrence relations in $D = 2$:

$$\omega_n^{l,\,T} = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \sum_{l'=0}^{2} \left( \delta^{ll'} - (-1)^n C_{su}^{ll'} \right) \left( \omega_k^{l',\,T} \omega_{n-k}^{l',\,T} + \omega_k^{l',\,R} \omega_{n-k}^{l',\,R} \right);$$

$$\omega_n^{l,\,R} = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \sum_{l'=0}^{2} \left( \delta^{ll'} - (-1)^n C_{st}^{ll'} \right) \left( \omega_k^{l',\,T} \omega_{n-k}^{l',\,R} + \omega_k^{l',\,R} \omega_{n-k}^{l',\,T} \right).$$

- Initial conditions ($n = 1$) from the tree-level calculation:

$$\omega_1^{l=0,\,T} = \omega_1^{l=0,\,R} = (2g_1(N+1) + g_2(N+3));$$
$$\omega_1^{l=1,\,T} = -\omega_1^{l=1,\,R} = (g_2 - 2g_1);$$
$$\omega_1^{l=2,\,T} = \omega_1^{l=2,\,R} = (2g_1 + 3g_2).$$

- For $n > 1$ only particular combination of couplings occurs in LL-coefficients:

$$1/G = 2g_1 + g_2.$$

# Diagonalization of the recurrence system I

- The system for $\omega_n^{l=0,\, T}$ is equivalent to

$$f_n = \frac{1}{n-1}\sum_{k=1}^{n-1}\Big(A_0 + (-1)^n A_1 + (-1)^k A_2\Big)\, f_k f_{n-k}; \quad f_1 = 1.$$

**The coefficients $A_i$ read:**

$$A_0 = 1 + \frac{1}{(N+2)(N-1)};\; A_1 = -\frac{N+1}{(N+2)(N-1)};\; A_2 = -\frac{2}{(N+2)(N-1)}.$$

- For $n \geq 2$:

$$\omega_n^{l=0,\, T} = f_n \times \left(\frac{(N+2)(N-1)}{NG}\right)^n.$$

# Diagonalization of the recurrence system II

- Generating function:

$$\Omega(z) = \sum_{n=2}^{\infty} \omega_n^{I=0,\ T} z^{n-1}.$$

- $I = 0$ transition amplitude in LL-approximation:

$$\mathcal{M}^{I=0,\ T}\Big|_{\rm LL}(s) = s^2\left[2g_1(N+1) + g_2(N+3)\right] + \frac{s^2}{G}\,\Omega\left(\frac{s}{2\pi G}\log\left(\frac{\mu^2}{s}\right)\right).$$

- All other $\mathcal{M}^{I,\ T,\ R}$ are expressed through the same $\Omega(z)$.

# How to solve the recurrence system I

- We introduce the generating function

$$f(z) = \sum_{n=1}^{\infty} f_n z^{n-1}.$$

- The recurrence system for $f_n$ is equivalent to the differential equation:

$$\frac{d}{dz} f(z) = A_0 \; f(z)^2 + A_1 \; f(-z)^2 - A_2 \; f(z) f(-z), \quad f(0) = 1.$$

- **N.b.** $A_1 = A_2 = 0$ same form as the RG-equation in a renormalizable QFT:

$$\frac{d}{dz} f(z) = A_0 \; f^2(z); \;\; f(z) = \frac{1}{1 - A_0 z} \; : \; \text{single Landau pole.}$$

# How to solve the recurrence system II

- Even and odd parts of the generating function: $f(\pm z) = u(z) \pm v(z)$

$$u(z) = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} f_n z^{n-1}; \quad v(z) = \sum_{\substack{n=2 \\ \text{even}}}^{\infty} f_n z^{n-1}.$$

- We need to address the system:

$$\begin{cases} v'(z) = (A_0 + A_1 - A_2)u^2(z) + (A_0 + A_1 + A_2)v^2(z); \\ u'(z) = 2(A_0 - A_1)u(z)v(z); \end{cases} \qquad u(0) = 1; \; v(0) = 0.$$

- The system possesses the following first integral A. Smirnov:

$$\left( u^2(z) + \frac{-A_0 + 3A_1 + A_2}{A_0 + A_1 - A_2} v^2(z) \right)^{A_0 - A_1} = (u(z))^{A_0 + A_1 + A_2}.$$

# How to solve the recurrence system III

- New variable:

$$I(z) = \frac{1}{u(z)}.$$

- Use of the first integral:

$$v(z) = -\frac{1}{2(A_0 - A_1)} \frac{1}{I(z)} \frac{d}{dz} I(z).$$

- Final form of the equation for $I(z)$:

$$\alpha_1 \left[ I'(z) \right]^2 = \alpha_0 I^{2\alpha_1}(z) - \alpha_0; \quad I(0) = 1.$$

$$\alpha_0 = -\frac{2N^3}{(N-1)^2(N+2)}; \quad \alpha_1 = \frac{N+2}{2N}.$$

# Equivalent mechanical system I

- The problem is equivalent to a 1D motion of a mechanical system:
  - "time": $t = \frac{N}{(N-1)(N+2)} z$;
  - "coordinate": $q(t) = I\left(\frac{(N-1)(N+2)}{N}\ t\right)$.

$$\frac{m\ \dot{q}(t)^2}{2} + q(t)^\gamma = 1; \quad q(t=0) = 1; \quad \dot{q}(t=0) = 0;$$

  - "mass": $m = \frac{1}{2N^2}$;
  - exponent of the "potential": $\gamma = \frac{N+2}{N}$.

**LL-amplitude through $q$**

$$\Omega(z) = \frac{(N-1)(N+2)}{N}\left(\frac{1}{q(z)} - 1\right) - \frac{N-1}{2N}\frac{d}{dz}\log(q(z)).$$

# Equivalent mechanical system II

- Dual "mechanical" system: $q(t) = r(t)^{\frac{2}{2-\gamma}}$:

$$\frac{M\,\dot{r}(t)^2}{2} + \left[2 - r(t)^{\delta}\right] = 1; \quad r(t=0) = 1; \quad \dot{r}(t=0) = 0;$$

  - "mass": $M = \frac{2}{(N-2)^2}$;
  - exponent of the "potential": $\delta = \frac{2\gamma}{\gamma-2}$.

## Remarkable solutions I

- $N \to \infty$; $(\gamma = 0,\ m \to 0)$ – "motion under constant force":

$$q(t) = 1 - (Nt)^2; \qquad \boxed{\Omega(z) = \frac{N}{1 - Nz} - N.}$$

**N.b.** LL-amplitude possesses a single Landau pole; assuming $g_i \sim 1/N$, theory is equivalent to a renormalizable QFT.

- $N = 2$ $(\gamma = 2,\ m = \frac{1}{8})$ – "harmonic oscillator":

$$q(t) = \cos(4t); \qquad \boxed{\Omega(z) = \frac{2}{\cos(4z)} + \tan(4z) - 2}.$$

First example of a quasi-renormalizable theory!

- Poles and residues:

$$z_k^{(1)} = \frac{\pi}{8}(4k+1),\ k \in \mathbb{Z}, \qquad \operatorname*{Res}_{z=z_k^{(1)}} \Omega(z) = -\frac{3}{4};$$

$$z_k^{(2)} = \frac{\pi}{8}(4k+3),\ k \in \mathbb{Z}, \qquad \operatorname*{Res}_{z=z_k^{(2)}} \Omega(z) = \frac{1}{4}.$$

## Remarkable solutions II

- $N \to 0$ $(\delta = 2,\ M = \frac{1}{2})$ – "inverted harmonic potential":

$$r(t) = \cosh(2t); \qquad \boxed{\Omega(z) = -\log[\cosh(2z)] - \tanh(2z)}.$$

Another example of a quasi-renormalizable theory!

- Poles and residues:

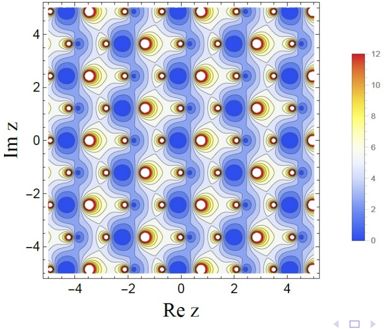
$$z_k = i\frac{\pi}{4}(2k+1),\ k \in \mathbb{Z}, \qquad \operatorname*{Res}_{z=z_k} \Omega(z) = -\frac{1}{2}.$$

## Remarkable solutions III

- Case $N \to 1$ ($\gamma = 3$, $m = \frac{1}{2}$) – (Bacher&Flajolet's pseudofactorial):

$$q(t) = \frac{3\wp\left(\sqrt{3}t; 0, -\frac{4}{27}\right) - 2}{3\wp\left(\sqrt{3}t; 0, -\frac{4}{27}\right) + 1}; \qquad \boxed{[\wp']^2 = 4\wp^3 - g_2\wp - g_3}$$

$$\frac{\Omega(z)}{N-1} = \frac{1}{2}\,\frac{6\wp\left(\sqrt{3}z; 0, -\frac{4}{27}\right) - \sqrt{3}\wp'\left(\sqrt{3}z; 0, -\frac{4}{27}\right) + 2}{\wp\left(\sqrt{3}z; 0, -\frac{4}{27}\right)^2 - \frac{1}{3}\wp\left(\sqrt{3}z; 0, -\frac{4}{27}\right) - \frac{2}{9}}.$$



## Qualitative general analysis

- Case $N > 0$ or $N < -2$: "particle" moves to the left from $q = 1$ to $q = 0$. This point is reached in finite "time" and with finite "velocity": pole singularity of $\Omega(z)$:

$$z_{\mathrm{pole}} = \frac{\sqrt{\pi}\Gamma\left(1 + \frac{N}{N+2}\right)}{2N\Gamma\left(\frac{1}{2} + \frac{N}{N+2}\right)}.$$

  - $N = \frac{2}{2p+1}$ ($p \in \mathbb{N}$) "particle" oscillates in the "potential" $q^{2(p+1)}$. $\Omega(z)$ has an infinite number of equidistant poles separated by:

$$\Delta z = \frac{(2p+1)\sqrt{\pi}\;\Gamma\left(1 + \frac{1}{2(p+1)}\right)}{2\;\Gamma\left(\frac{1}{2} + \frac{1}{2(p+1)}\right)}, \quad p \in \mathbb{N}.$$

  - Known elliptic solutions for $N = \frac{2}{3}, \frac{2}{5}$.

- Case $-2 < N < 0$: "particle" moves to the right from $q = 1$ and never reaches $q = 0$. No singularities on the real axis for $\Omega(z)$.

$$z_{\mathrm{branch}} = \pm i \frac{\sqrt{\pi}\Gamma\left(\frac{2}{N+2} - \frac{1}{2}\right)}{2N\;\Gamma\left(-\frac{N}{N+2}\right)}.$$

# $\mathrm{O}(N+1)$-symmetric $\sigma$-model deformed beyond $T\bar{T}$

- $\mathrm{O}(N+1)$-symmetric $\sigma$-model deformed by the most general dimension-4 operator:

$$S=\int d^2x\left(\frac{1}{2}\partial_\mu\vec{n}\cdot\partial^\mu\vec{n}-g_1\left(\partial_\mu\vec{n}\cdot\partial^\mu\vec{n}\right)\left(\partial_\nu\vec{n}\cdot\partial^\nu\vec{n}\right)-g_2\left(\partial_\mu\vec{n}\cdot\partial_\nu\vec{n}\right)\left(\partial^\mu\vec{n}\cdot\partial^\nu\vec{n}\right)\right)$$

- $\vec{n}\in S^N$: $\vec{n}=\left(\Phi^1,\ldots\Phi^N,\sqrt{F^2-\vec{\Phi}^2}\right)$; $\quad [F]=0;\quad [g_1]=[g_2]=-2.$

$$S=\int d^2x\left(\frac{1}{2}\partial_\mu\vec{n}\cdot\partial^\mu\vec{n}-4\lambda\det\left(T_{\mu\nu}\right)-\frac{1}{G}\left(\partial_\mu\vec{n}\cdot\partial_\nu\vec{n}\right)\left(\partial^\mu\vec{n}\cdot\partial^\nu\vec{n}\right)\right)$$

$$T_{\mu\nu}=\partial_\mu\vec{n}\cdot\partial_\nu\vec{n}-\frac{1}{2}\eta_{\mu\nu}\left(\partial_\alpha\vec{n}\cdot\partial^\alpha\vec{n}\right)$$

$$\lambda=-g_1;\quad \frac{1}{G}=2g_1+g_2.$$

# Leading chiral logs in O($N$)-symmetric mixed theory in $D = 2$

- $m$ counts powers of renormalizable interaction with constant $\frac{1}{F^2}$
- $n$ counts powers of non-renormalizable interaction (with the constant $\frac{1}{G} = 2g_1 + g_2$ for $n + m \geq 2$).
- $2 \to 2$ amplitude in LL approximation:

$$\mathcal{M}(s)^{I=0,\{T,R\}}\Big|_{\mathrm{LL}} = \frac{s}{F^2}(N-1) + s^2\left(2g_1(N+1) + g_2(N+3)\right)$$
$$\underbrace{+\,4\pi s \sum_{\substack{n,m=0\\ n+m\geq 2}}^{+\infty} \left(\frac{s}{4\pi G}\right)^n \left(\frac{1}{4\pi F^2}\right)^m \omega_{n,m}^{I,\{T,R\}} \left[\log\left(\frac{\mu^2}{s}\right)\right]^{n+m-1}}_{\frac{s^2}{G}\Omega\left(\frac{s}{4\pi G}\log\left(\frac{\mu^2}{s}\right),\,\frac{G}{sF^2}\right)} + \mathcal{O}(NLL)$$

- Tree level result $m = 1$, $n = 0$ and $m = 0$, $n = 1$ are given in Red.
- Generating function:

$$\Omega(z, w) = \sum_{\substack{n,m=0\\ n+m\geq 2}}^{+\infty} \omega_{n,m}^{I=0,T} z^{n+m-1} w^m$$

# Recurrence relations for the mixed theory I

- Rescaling of LL-coefficients:

$$\omega^{0,T}_{n,m} = f_{n,m} \left( \frac{(N-1)(N+2)}{N} (2g_1 + g_2) \right)^n (N-1)^m$$

- Equivalent recurrence relation:

$$f_{n,m} = \frac{1}{n+m-1} \sum_{k=0}^{n} \sum_{l=0}^{m} \underbrace{\left( A_0 + (-1)^n A_1 + (-1)^k A_2 \right)}_{A(n,k)} f_{k,l} f_{n-k,m-l}$$

- Same $A(n,k)$ as for the bi-quartic theory (no dependence on $m$):

$$A_0 = 1 + \frac{1}{(N+2)(N-1)}; \quad A_1 = -\frac{N+1}{(N+2)(N-1)}; \quad A_2 = -\frac{2}{(N+2)(N-1)};$$

- The initial conditions:

$$f_{0,0} = 0, \;\; f_{1,0} = f_{0,1} = 1.$$

## Recurrence relations for the mixed theory II

- The recurrence system look unassailable. But!
- Generating functions of definite combined parity (with respect to $z$ and $w$):

$$u(-z,-w) = f(z,w); \quad v(-z,-w) = -g(z,w).$$

$$u(z,w) = \sum_{\substack{n=1 \\ \mathrm{odd}}}^{\infty} \sum_{\substack{m=0 \\ \mathrm{even}}}^{\infty} f_{n,m} z^{n+m-1} w^m + \sum_{\substack{n=0 \\ \mathrm{odd}}}^{\infty} \sum_{\substack{m=1 \\ \mathrm{odd}}}^{\infty} f_{n,m} z^{n+m-1} w^m;$$

$$v(z,w) = \sum_{\substack{n=0 \\ \mathrm{even}}}^{\infty} \sum_{\substack{m=1 \\ \mathrm{odd}}}^{\infty} f_{n,m} z^{n+m-1} w^m + \sum_{\substack{n=0 \\ \mathrm{even}}}^{\infty} \sum_{\substack{m=0 \\ \mathrm{even}}}^{\infty} f_{n,m} z^{n+m-1} w^m; \quad (\textbf{N.b.}\ f_{0,0} = 0).$$

- The $\Omega$ generating function (after rescaling):

$$u(z,w) + v(z,w) \to \Omega(z,w)$$

# Recurrence relations for the mixed theory III

- The recurrence system is equivalent to:

$$\begin{cases} \frac{\partial}{\partial z} v(z,w) = (A_0 + A_1 - A_2) u^2(z,w) + (A_0 + A_1 + A_2) v^2(z,w); \\ \frac{\partial}{\partial z} u(z,w) = 2(A_0 - A_1) u(z,w) v(z,w). \end{cases}$$

- The initial conditions read

$$u(z=0,w) = 1; \quad v(z=0,w) = w.$$

- The dependence on $w$ enters only through the initial condition!
- Same system as for bi-quartic theory but with different initial condition!
- First integral:

$$\left( \frac{u^2(z,w) - \frac{(N+2)^2}{N^2} v^2(z,w)}{1 - \frac{(N+2)^2}{N^2} w^2} \right)^{\frac{N}{N-1}} = u^{\frac{N-2}{N-1}}(z,w).$$

## Equivalent mechanical system

- Rescaled variables: "time" and "coordinate":

$$t = \frac{N}{(N-1)(N+2)} z; \qquad q(t,w) = \frac{1}{u\left(\frac{(N-1)(N+2)}{N} t, \frac{N}{(N+2)} w\right)}.$$

- The equation of motion

$$\frac{m}{2} \dot{q}^2(t,w) + \left(1 - w^2\right) q(t,w)^{\gamma} = 1; \quad q(t=0,w) = 1.$$

  - "Particle" mass $m = \frac{1}{2N^2}$.
  - The "potential" $\left(1 - w^2\right) q^{\gamma}(t,w)$ with $\gamma = \frac{N+2}{N}$
  - The total energy: $E = 1$.

- N.b. The initial condition implies non-zero initial velocity:

$$\dot{q}^2(t=0,w) = \mathrm{v}_0^2 = 4N^2 w^2.$$

## Solution for pure $T\bar{T}$ deformed 2D $\sigma$-model

- Pure $T\bar{T}$ deformation corresponds to $G \to \infty$
- The loop function:

$$\Omega^{T\bar{T}}(y) = \lim_{\substack{w\to\infty \\ z\to 0,\, wz=y}} \frac{1}{w}\Omega(z,w) = (N-1)\frac{(N-2)y}{1-(N-2)y}.$$

- Transition $I=0$ amplitude in the LL approximation:

$$\mathcal{M}^{I=0,\,T}(s) = 4\lambda s^2 + \frac{s}{F^2}(N-1)\left(\frac{1}{1-\frac{(N-2)}{4\pi F^2}\log\left(\frac{\mu^2}{s}\right)}\right)$$
$$= 4\lambda s^2 + \frac{s}{F^2(s)}(N-1).$$

- The running coupling:

$$\frac{d\,\frac{1}{F^2(s)}}{d\log\mu^2} \equiv \beta\left(\frac{1}{F^2}\right) = \frac{N-2}{4\pi}\frac{1}{F^4} + \mathcal{O}\left(\frac{1}{F^6}\right).$$

- N.b. The $\beta$-function contains the factor $N-2$ instead of $N-1$. Why?

# A word about $O(N+1)/O(N)$ sigma-model in $D=2$

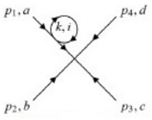
$$\mathcal{L} = \frac{1}{2g^2} |\partial_\mu \vec{n} \cdot \partial^\mu \vec{n}| ; \quad \vec{n}\cdot\vec{n} = 1; \qquad \vec{n} = \left(\pi^1, \ldots, \pi^N, \sigma\right); \quad \sigma = \sqrt{1-\vec{\pi}^2}.$$

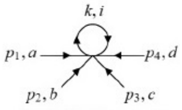
$$\mathcal{L} = \frac{1}{2g^2} |\partial_\mu \vec{\pi}|^2 + \frac{1}{2g^2} (\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 + \ldots .$$

- Asymptotically free theory in $D=2$ A.M. Polyakov, 1975:

$$\beta(g) = -(N-1)\frac{g^3}{4\pi} + O(g^5); \qquad \beta(g(\mu)) \equiv \frac{\partial}{\partial \log \mu} g(\mu).$$

- The logs we sum with help of the recurrence relations are $\log s$, not $\log \mu$!
- Reason: infrared divergencies. Unitarity method misses $\log \frac{m}{\mu}$, where $m$ is the small infrared regulating mass.
- Diagrams we miss (at 1-loop):



## Solutions for $N = 2$ case I

- LL-amplitude $\Omega(z, w)$ through $q(z, w)$

$$\Omega(z,w) = \frac{(N+2)(N-1)}{N}\left(\frac{1}{q(z,w)} - 1 - \frac{N}{N+2}w\right) - \frac{N-1}{2N}\frac{d}{dz}\log(q(z,w))$$

- The analytic structure of the solution strongly depends on $w$:

**N.b.** $w \to \frac{G}{sF^2}$ : relative strength of the two interactions.

- $\boxed{|w| > 1 : w = \pm \operatorname{ch}(\alpha) \text{ with } \alpha \in [0, \infty]}$

$$\Omega(z,w) = \frac{2\operatorname{sh}(\alpha)}{\operatorname{sh}(\alpha \mp 4z\operatorname{sh}(\alpha))} + \frac{\operatorname{sh}(4z\operatorname{sh}(\alpha))}{\operatorname{sh}(\alpha \mp 4z\operatorname{sh}(\alpha))} - 2$$

$\Omega(z, w)$ possesses complex poles.

- $\boxed{w = 1 :}$

$$q(t,w) = 1 - 4t$$

$\Omega(z, w)$ possesses real pole at $z = \frac{1}{4}$.

# Solutions for $N = 2$ case II

- $-1 < w < 1 : w = \sin(\alpha)$ with $\alpha \in [-\pi/2, \pi/2]$.

$$\Omega(z, w) = \frac{2\cos(\alpha)}{\cos(\alpha + 4z\cos(\alpha))} + \frac{\sin(4z\cos(\alpha))}{\cos(\alpha + 4z\cos(\alpha))} - 2$$

$\Omega(z, w)$ possesses real poles. N.b. For $w = 0$ ($F \to \infty$) $\alpha = 0$ and biquartic model is reproduced.

- $w = -1 :$

$$q(t, w) = 1 + 4t$$

$\Omega(z, w)$ possesses real pole at

$$z = -\frac{1}{4}.$$

## More solutions

| $N$ | $\gamma$ | $m$ | $\Omega(z,w)$ | Comments: |
|---|---|---|---|---|
| $-2$ | 0 | $\frac{1}{8}$ | 0 | free motion; no LLs; |
| $N \to \infty$ | 1 | $m \to 0$ | ✓ | motion under constant force; |
| 2 | 2 | $\frac{1}{8}$ | ✓ | harmonic oscillator case; trigonometric solution; |
| $N \to 1$ | 3 | $\frac{1}{2}$ | ✓ | Dixon's functions elliptic solution; |
| $\frac{2}{3}$ | 4 | $\frac{9}{8}$ | ✓ | elliptic solution; |

**Table :** Summary of solutions for the mechanical system expressed in terms of elliptic functions (and their degeneracies).

| $N$ | $\delta$ | $M$ | $\Omega(z,w)$ | Comments: |
|---|---|---|---|---|
| $-2$ | 0 | $\frac{1}{8}$ | 0 | $\gamma = 0$ case; |
| $-\frac{2}{3}$ | 1 | $\frac{9}{32}$ | ✓ | motion under constant force; |
| $N \to 0$ | 2 | $\frac{1}{2}$ | ✓ | motion in the inverted harmonic potential; |
| $\frac{2}{5}$ | 3 | $\frac{25}{32}$ | ✓ | elliptic solution; |
| $\frac{2}{3}$ | 4 | $\frac{9}{8}$ | ✓ | $\gamma = 4$ case; |

**Table :** Summary of solutions for the dual mechanical system $q(t,w) = r(t,w)^{\frac{2}{2-\gamma}}$ ($M = \frac{2}{(N-2)^2}$; $\delta = \frac{2\gamma}{\gamma-2}$) expressed in terms of elliptic functions (and their degeneracies).

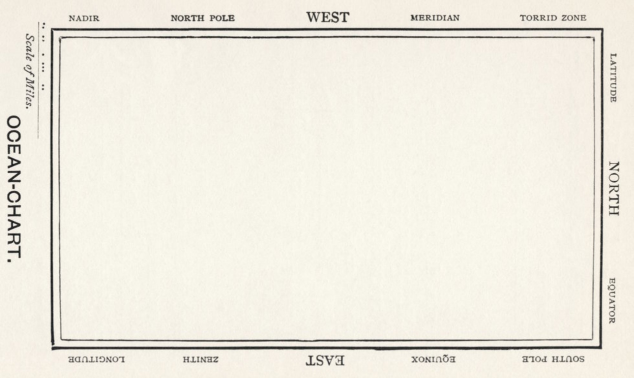
# Conclusions and Outlook

1. First examples of quasi-renormalizable theories:
   - All order summation of LLs in the O(2)-symmetric free theory deformed beyond $T\bar{T}$ in $D = 2$ ($z = \frac{s}{4\pi G} \log\left(\mu^2/s\right)$) :

$$\Omega^{N=2}(z) = \frac{2}{\cos(4z)} + \tan(4z) - 2.$$

   - $N \to 1$ limit case. LL-amplitude in terms of Dixon's functions.
2. All order summation of LLs in the O(3)/O(2)-symmetric $\sigma$-model deformed beyond $T\bar{T}$ in $D = 2$: extremely rich dynamics.
3. Implications for 2-point function? Landau ghosts and their physical interpretation?
4. Infrared problem?
5. Any realistic physical systems to be described by the mixed theory in $D = 2$? Connections to solid state physics? Phase transitions?
6. Is it possible to have a non-trivial example of the exactly solvable quasi-renormalizable theory in $D = 2$?

## Where we are?




*He had bought a large map representing the sea,*
*Without the least vestige of land:*
*And the crew were much pleased when they found it to be*
*A map they could all understand.*

# Where we actually are