Comments on $T\overline{T}$–deformed non-relativistic models

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APCTP workshop "Exact results on irrelevant deformations of QFTs"

Outline

1. Introduction
2. Action
3. Examples
4. Comments I
5. Solitons
6. Comments II
7. Open questions

$T\overline{T}$ deformation

Zamolodchikov '04

$$S = \int \mathrm{d}x\mathrm{d}t\, \mathcal{L}\,, \qquad \frac{\partial \mathcal{L}}{\partial \alpha} = -T\overline{T}$$

$$T\overline{T} \equiv {T^t}_x {T^x}_t - {T^x}_x {T^t}_t = \epsilon_{\mu\nu} {T^\mu}_t {T^\nu}_x = \det T_{\mu\nu}$$

The Lagrangian density of a "seed" theory

$$\mathcal{L}_0 \equiv \mathcal{L}\,|_{\alpha=0}$$

- $T\overline{T}$ as a series in α is a well-defined operator
- Irrelevant deformation

Properties of the $T\overline{T}$ deformation

Zamolodchikov '04

If the seed model is Lorentz invariant and renormalisable, and $T_{\mu\nu}$ is symmetric

- Spectrum obeys the inhomogeneous inviscid Burgers eq

$$\partial_\alpha \mathcal{E}_\alpha(R,\mathcal{P}) + \frac{1}{2}\partial_R(\mathcal{E}_\alpha^2(R,\mathcal{P}) - \mathcal{P}^2) = 0\,, \quad \mathcal{P} = \frac{2\pi k}{R}$$

- Two-particle S-matrices are related as

$$S(\theta) = e^{-i\alpha m^2 \sinh\theta} S^{(0)}(\theta)\,, \quad \theta = \theta_1 - \theta_2$$

- If the seed model is integrable then the $T\overline{T}$ deformed model is also integrable

Smirnov, Zamolodchikov '16 & Cavaglià, Negro, Szécsényi, Tateo '16

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Connections to two-dimensional gravity

- A $T\overline{T}$ deformed S-matrix and the partition function can be obtained by coupling a seed model to the flat space Jackiw-Teitelboim gravity Dubovsky, Gorbenko, Mirbabayi '17
- This leads to the interpretation of the $T\overline{T}$ deformation as a nonlocal field dependent change of space-time coordinates of the seed model Conti, Negro, Tateo '18
- The partition function of a deformed model can be derived by coupling a seed model to a random geometry Cardy '18
- The action of a $T\overline{T}$ deformed model can be obtained by interpreting it as the action of a non-critical string sigma model in a light-cone gauge Baggio, Sfondrini, Tartaglino-Mazzucchelli, Walsh '18 & Frolov '19
- The action for $T\overline{T}$ deformations with the canonical stress-energy tensor is universal Frolov '19

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In this talk

- Derive the $T\overline{T}$ deformed actions for sigma-model, the matrix NLS equation and the Gardner equation by using the universal action Esper, Frolov '21
- Analyse solitons of the deformed NLS and KdV models to see whether they exhibit the phenomenon of widening/narrowing the width of particles under the $T\overline{T}$ deformation Cardy, Doyon '20
- The $T\overline{T}$ deformed action for the (non-matrix) NLS model was also found by using different and substantially more complicated methods Hansen, Jiang, Xu '20 & Ceschin, Conti, Tateo '20 & Chen, Hou, Tian '20
- Deformed soliton solutions were also analysed Ceschin, Conti, Tateo '20

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Universal $T\overline{T}$ deformed action

Frolov '19

Seed model with the action

$$S_0 = \int \mathrm{d}x\mathrm{d}t\, \mathcal{L}_0\,, \quad \mathcal{L}_0 = P^t_a(\Psi)\partial_t\Psi^a + P^x_a(\Psi)\partial_x\Psi^a - \mathrm{V}(\Psi)$$

- $\Psi^a\,,\ a = 1,\dots,n$ are bosonic and fermionic fields which can be real or complex
- If a field is complex then the set (Ψ^a) also includes its complex conjugate field
- P^t_a, P^x_a and V are such that the action is real and Grassmann even but otherwise they are arbitrary functions of Ψ^a
- The seed action is written in the first-order formalism wrt both time and space $\Rightarrow$ many of the fields are non-dynamical
- If each Ψ^a belongs to a Lorentz group representation and P^t_a, P^x_a belong to the conjugate representation, and V is a Lorentz scalar then the seed model is Lorentz invariant.

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The light-cone gauge approach to $T\overline{T}$ deformed models leads to

$$\mathcal{L} = \frac{\mathrm{K}_t^t + \mathrm{K}_x^x - \mathrm{V} + \alpha(\mathrm{K}_t^t \mathrm{K}_x^x - \mathrm{K}_x^t \mathrm{K}_t^x)}{1+\alpha \mathrm{V}} = \frac{\mathcal{L}_0 - \frac{\alpha}{2}\epsilon^{\gamma\rho}\epsilon_{\mu\nu}\mathrm{K}_\gamma^\mu \mathrm{K}_\rho^\nu}{1+\alpha \mathrm{V}}$$

- $\mathrm{K}_\gamma^t \equiv P_a^t \partial_\gamma \Psi^a\,, \quad \mathrm{K}_\gamma^x \equiv P_a^x \partial_\gamma \Psi^a\,, \quad \gamma = t, x$
- The Levi-Civita symbol is defined by $\epsilon^{01} = \epsilon^{tx} = 1 = \epsilon_{xt} = \epsilon_{10}$
- The canonical stress-energy tensor is

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \Psi^a}\partial_\nu \Psi^a - \delta^\mu_\nu \mathcal{L}$$

$$T^t{}_t = \frac{-\mathrm{K}_x^x + \mathrm{V}}{1+\alpha\mathrm{V}}\,, \quad T^x{}_t = \frac{\mathrm{K}_t^x}{1+\alpha\mathrm{V}}\,, \quad T^t{}_x = \frac{\mathrm{K}_x^t}{1+\alpha\mathrm{V}}\,, \quad T^x{}_x = \frac{-\mathrm{K}_t^t + \mathrm{V}}{1+\alpha\mathrm{V}}$$

- $\mathcal{L}$ satisfies the flow equation

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- Any seed model can be written in the first-order form $\Rightarrow$ the $T\overline{T}$ deformed Lagrangian is universal
- In a non-relativistic case $\mathcal{L}_0$ may include total derivative terms
- They change the canonical stress-energy tensor
- The Lagrangian and the equations of motion of the deformed model depend on the total derivative terms
- This dependence probably cannot be undone by a field redefinition

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$T\overline{T}$ deformed sigma model

Esper, Frolov '21

Sigma-model of n scalar fields

$$\mathcal{L}_0 = \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j \, G_{ij}(X) + \frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j \, B_{ij}(X) - U(X)$$

$\eta^{\alpha\beta} = \mathrm{diag}(1,-1)$, $\epsilon^{01} = \epsilon^{tx} = 1 = \epsilon_{xt}$, U is an arbitrary potential

- Introduce the momentum vectors

$$P_i^\alpha = \frac{\partial \mathcal{L}_0}{\partial \partial_\alpha X^i} = (\eta^{\alpha\beta} G_{ij} + \epsilon^{\alpha\beta} B_{ij})\partial_\beta X^j \qquad (3.1)$$

- Rewrite $\mathcal{L}_0$ in the first-order formalism

$$\mathcal{L}_0 = P_i^\gamma \partial_\gamma X^i - \frac{1}{2}(\eta_{\gamma\rho}\widetilde{G}^{ij} + \epsilon_{\gamma\rho}\widetilde{B}^{ij}) P_i^\gamma P_j^\rho - U$$

- $\widetilde{G}^{ij}$ and $\widetilde{B}^{ij}$ satisfy

$$\widetilde{G}^{ij}(G_{jk} - B_{jl}G^{lm}B_{mk}) = \delta^i_k\,, \quad \widetilde{B}^{ij} = -\widetilde{G}^{ik}B_{kl}G^{lj} = -G^{ik}B_{kl}\widetilde{G}^{lj}$$

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- The set (Ψ^a) consists of X^i, and P_i^γ, and

$$\mathrm{K}_t^t = P_i^t \partial_t X^i\,, \quad \mathrm{K}_x^x = P_i^x \partial_x X^i\,, \quad \mathrm{K}_x^t = P_i^t \partial_x X^i\,, \quad \mathrm{K}_t^x = P_i^x \partial_t X^i$$

$$V = \frac{1}{2}\big(\eta_{\gamma\rho}\,\widetilde{G}^{ij} + \epsilon_{\gamma\rho}\,\widetilde{B}_{ij}\big) P_i^\gamma P_j^\rho + U$$

- The $T\overline{T}$ deformed Lagrangian of the sigma model is

$$\mathcal{L} = \frac{P_i^\gamma \partial_\gamma X^i - \frac{1}{2}\big(\eta_{\gamma\rho}\,\widetilde{G}^{ij} + \epsilon_{\gamma\rho}\,\widetilde{B}^{ij}\big) P_i^\gamma P_j^\rho - U - \frac{\alpha}{2}\epsilon^{\gamma\rho}\epsilon_{\mu\nu} P_i^\mu \partial_\gamma X^i P_j^\nu \partial_\rho X^j}{1 + \frac{\alpha}{2}\big(\eta_{\gamma\rho}\,\widetilde{G}^{ij} + \epsilon_{\gamma\rho}\,\widetilde{B}_{ij}\big) P_i^\gamma P_j^\rho + \alpha U}$$

$T\overline{T}$ deformed sigma model

Esper, Frolov '21

$$\mathcal{L} = \frac{P_i^\gamma \partial_\gamma X^i - \frac{1}{2}\left(\eta_{\gamma\rho}\,\widetilde{G}^{ij} + \epsilon_{\gamma\rho}\,\widetilde{B}^{ij}\right) P_i^\gamma P_j^\rho - U - \frac{\alpha}{2}\epsilon^{\gamma\rho}\epsilon_{\mu\nu} P_i^\mu \partial_\gamma X^i P_j^\nu \partial_\rho X^j}{1 + \frac{\alpha}{2}\left(\eta_{\gamma\rho}\,\widetilde{G}^{ij} + \epsilon_{\gamma\rho}\,\widetilde{B}_{ij}\right) P_i^\gamma P_j^\rho + \alpha U}$$

- One can get rid of the auxiliary fields P_i^γ and get

$$\mathcal{L}_{\rm ph} = -\frac{1}{\alpha} + \frac{1}{2\tilde{\alpha}} + \frac{1}{2\tilde{\alpha}}\sqrt{1 + 2\tilde{\alpha}(\dot{X}^2 - X'^2) - 4\tilde{\alpha}^2(\dot{X}^2 X'^2 - (\dot{X}X')^2)} + \dot{X}^i X'^j B_{ij}$$

$$\dot{X}^2 \equiv G_{ij}\dot{X}^i\dot{X}^j\,, \quad X'^2 \equiv G_{ij}X'^i X'^j\,, \quad \dot{X}X' \equiv G_{ij}\dot{X}^i X'^j\,, \quad \tilde{\alpha} = \alpha(1 + \alpha U)$$

- $\mathcal{L}_{\rm ph}$ admits perturbative expansion in α
- $\mathcal{L}$ describes both perturbative and non-perturbative in α solutions of the eom

$T\bar{T}$ deformed matrix nonlinear Schrödinger model

Esper, Frolov '21

$$\mathcal{L}_0 = \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \bar{\psi}'\psi' - U\,, \quad U = \kappa\,\bar{\psi}\psi\bar{\psi}\psi - \mu\,\bar{\psi}\psi$$

- ψ and $\bar{\psi}$ are complex $n \times m$ and $m \times n$ matrices hermitian conjugate to each other.

$$\psi = (\psi_{ai})\,, \quad \bar{\psi} = \psi^\dagger = (\psi^*_{ia})\,, \quad a = 1, \dots, n\,,\ i = 1, \dots, m$$

- The trace is implied, i.e.

$$\bar{\psi}\dot{\psi} \equiv \psi^*_{ia}\dot{\psi}_{ai}\,, \quad \bar{\psi}\psi\bar{\psi}\psi \equiv \psi^*_{ia}\psi_{aj}\psi^*_{jb}\psi_{bi}$$

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- Introduce

$$A = (A_{ai})\,, \quad \bar A = A^\dagger = (A^*_{ia})\,, \quad a = 1,\dots,n\,,\ i = 1,\dots,m$$

- Rewrite $\mathcal{L}_0$

$$\mathcal{L}_0 = \frac{i}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi) - \bar A\psi' - \bar\psi' A + \bar A A - U$$

- The set (Ψ^a) consists of $\psi, \bar\psi, A, \bar A$, and

$$\mathrm{K}^t_t = \frac{i}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi)\,, \quad \mathrm{K}^x_x = -\bar A\psi' - \bar\psi' A\,,$$

$$\mathrm{K}^t_x = \frac{i}{2}(\bar\psi\psi' - \bar\psi'\psi)\,, \quad \mathrm{K}^x_t = -\bar A\dot\psi - \dot{\bar\psi}A\,, \quad V = U - \bar A A$$

the trace is agian implied.

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$$A = (A_{ai})\,, \quad \bar A = A^\dagger = (A^*_{ia})\,, \quad a = 1,\dots,n\,,\ i = 1,\dots,m$$

- Rewrite $\mathcal{L}_0$

$$\mathcal{L}_0 = \frac{i}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi) - \bar A\psi' - \bar\psi' A + \bar A A - U$$

- The set (Ψ^a) consists of $\psi, \bar\psi, A, \bar A$, and

$$\mathrm{K}_t^t = \frac{i}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi)\,, \quad \mathrm{K}_x^x = -\bar A\psi' - \bar\psi' A\,,$$
$$\mathrm{K}_x^t = \frac{i}{2}(\bar\psi\psi' - \bar\psi'\psi)\,, \quad \mathrm{K}_t^x = -\bar A\dot\psi - \dot{\bar\psi} A\,, \quad V = U - \bar A A$$

the trace is agian implied.

$T\overline{T}$ deformed matrix nonlinear Schrödinger model

Esper, Frolov '21

The $T\overline{T}$ deformed Lagrangian of the matrix NLS

$$\mathcal{L} = \frac{K^t_t - \bar{A}\psi' - \bar{\psi}'A + \bar{A}A - U - \alpha\left(K^t_t(\bar{A}\psi' + \bar{\psi}'A) - K^t_x(\bar{A}\dot{\psi} + \dot{\bar{\psi}}A)\right)}{1 - \alpha(\bar{A}A - U)}$$

- Eliminating $A, \bar{A}$, one gets

$$\mathcal{L}_{\rm ph} = -\frac{1}{\alpha} + \frac{1 + \alpha K^t_t + \sqrt{\Lambda}}{2\tilde{\alpha}}\,, \quad \tilde{\alpha} = \alpha(1 + \alpha U)$$

$$\Lambda = (1 + \alpha K^t_t)^2(1 - 4\tilde{\alpha}\bar{\psi}'\psi') + 4\alpha\tilde{\alpha}(1 + \alpha K^t_t)K^t_x(\dot{\bar{\psi}}\psi' + \bar{\psi}'\dot{\psi}) - 4\alpha^2\,\tilde{\alpha}\,(K^t_x)^2\,\dot{\bar{\psi}}\dot{\psi}$$

  In Λ the trace is implied.

- The Poisson structure is modified
- Developing a Hamiltonian formulation requires dealing with an intricate system of second-class constraints

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$T\overline{T}$ deformed Gardner equation

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The Gardner equation is a combined KdV-mKdV equation

$$\dot{u} + \mu\, u' + 6\, g\, uu' - 6\, h\, u^2 u' + u''' = 0$$

- g, h and μ are constants
- If u satisfies periodic boundary conditions then μ can be removed by a constant shift of u

$$u \to u - c\,, \quad hc^2 + gc - \frac{\mu}{6} = 0$$

  which also changes g
- For decreasing boundary conditions such a shift is forbidden
- The Gardner equation is the continuity equation for the current

$$J^t = u\,, \quad J^x = \mu\, u + 3\, g\, u^2 - 2\, h\, u^3 + u''$$

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$$\dot{u} + \mu\, u' + 6\, g\, uu' - 6\, h\, u^2 u' + u''' = 0$$

The Gardner equation can be derived from the action

$$S_0 = \int \mathrm{d}x\mathrm{d}t\, \mathcal{L}_0\,, \quad \mathcal{L}_0 = \kappa\,(-\dot{\phi}\phi' - \mu\phi'^2 - 2g\phi'^3 + h\phi'^4 + \phi''^2\,)\,,$$

- κ is any constant, ϕ satisfies the boundary conditions
$$\phi(t,\infty) - \phi(t,-\infty) = Q_\phi = \text{const}$$
- u is related to ϕ as $u = \phi'$
- In the undeformed case $Q_\phi = Q$
- Eom for ϕ is invariant under a shift of ϕ by any function of time.
- By using this invariance one may require $\phi(t,\pm\infty) = \text{const}$
- In the deformed case this invariance is broken, and different time dependence of $\phi(t,\infty)$ leads to different solutions

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$$S_0 = \int \mathrm{d}x\mathrm{d}t\, \mathcal{L}_0\,, \quad \mathcal{L}_0 = \kappa\,(-\dot{\phi}\phi' - \mu\phi'^2 - 2g\phi'^3 + h\phi'^4 + \phi''^2\,)$$

- Introduce three auxiliary fields $\mathfrak{u}\,,\ A\,,\ B$, and cast $\mathcal{L}_0$ into

$$\mathcal{L}_0 = \kappa\,(-\mathfrak{u}\,\dot{\phi} - B\phi' + 2A\,\mathfrak{u}' + \mathfrak{u}\,B - \mu\,\mathfrak{u}^2 - 2g\,\mathfrak{u}^3 + h\,\mathfrak{u}^4 - A^2)$$

- $\mathfrak{u}$ is the Gardner field u
- $\mathcal{L}_0$ is invariant under constant shifts of ϕ $\Rightarrow$

$$J^t = u\,, \quad J^x = B\,, \quad \partial_\mu J^\mu = 0$$

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$$\mathcal{L}_0 = \kappa\left(-\mathrm{u}\,\dot\phi - B\phi' + 2A\,\mathrm{u}' + \mathrm{u}\,B - \mu\,\mathrm{u}^2 - 2g\,\mathrm{u}^3 + h\,\mathrm{u}^4 - A^2\right)$$

- The set (Ψ^a) consists of ϕ, u, B, A, and

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- $\mathcal{L}_0$ changes under the transformation

$$\phi \to \phi + f(t)\,, \quad B \to B + \frac{df}{dt}$$

by a derivative term

$$\mathcal{L}_0 \to \mathcal{L}_0 - \kappa \frac{\partial}{\partial x}\Big(\frac{df}{dt}\phi\Big)$$

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Similarities and differences

- All the Lagrangians depend on auxiliary fields
- If the physical fields of a seed model do not depend on second- or higher-order derivatives then auxiliary fields
  - enter a $T\overline{T}$ deformed Lagrangian algebraically
  - can be eliminated leading in the cases considered to Nambu-Goto type actions
- More complicated seed models (even relativistic invariant) may lead to $T\overline{T}$ deformed Lagrangians which are solutions to high degree polynomial equations

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Similarities and differences

A Nambu-Goto type Lagrangian obtained by eliminating auxiliary fields has a square root sign ambiguity

- If a model is on a line then finiteness of the energy singles out the perturbative in α branch
- If the model is on a circle then ???
- Consider the $T\overline{T}$ deformed free massless scalars and choose the negative sign in front of the square root in the $T\overline{T}$ deformed Lagrangian
- For $\alpha < 0$ the energy is not bounded from below
- If $\alpha > 0$ then the energy of any solution is bounded from below, and diverges in the limit $\alpha \to 0$
- Should the contribution from the nonperturbative branch be included in, for example, the partition function?
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The $T\overline{T}$ deformation drastically modifies the Poisson structure of all the non-relativistic models

- Developing a Hamiltonian formulation requires dealing with an intricate system of second-class constraints
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- This also makes unclear how to derive an analog of the inhomogeneous inviscid Burgers eq

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Quantum $T\overline{T}$ deformed non-relativistic models?

Do $T\overline{T}$ deformed non-relativistic models exist as quantum theories?

- No principal difficulties in perturbative quantisation. The expansion in powers of α is straightforward, and the standard technique can be used to compute the scattering matrix
- S-matrices would differ only by the $T\overline{T}$ CDD factor
- The relation between the S-matrices should be considered as a part of the definition of a quantised $T\overline{T}$ deformed model
- The spectrum of the $T\overline{T}$ deformed NLS (and LL) model on a circle can be also studied perturbatively
- At each order in α one can remove all interaction terms with time derivatives of ψ by a field redefinition producing new terms with higher space derivatives
- The resulting model has the undeformed Poisson structure and can be easily quantised. The spectrum of the Hamiltonian can then be found as an expansion in powers of α

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For finite α a more pragmatic approach to the $T\overline{T}$ deformed spectrum is to postulate that it is governed by the usual Bethe equations with the $T\overline{T}$ deformed S-matrix

- Done for the deformed NLS model in the repulsive regime Jiang '20
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Undeformed NLS soliton

- The undeformed one-soliton solution exists for $\kappa < 0$

$$\kappa = -\frac{g^2}{4}\,, \quad g > 0$$

- The undeformed one-soliton solution

$$\psi = \frac{u}{g}\,\frac{1}{\cosh\left(\frac{u}{2}(x - vt)\right)}\,e^{i\phi}\,, \quad \phi = \frac{v}{2}(x - vt) + \frac{t}{4}\left(u^2 + v^2 + 4\mu\right)$$

- U(1) charge Q, momentum P and energy E of the soliton

$$Q = \int_{-\infty}^{\infty} dx\,\bar{\psi}\psi = \frac{4u}{g^2}\,,$$

$$P = -\int_{-\infty}^{\infty} dx\,T^t{}_x = \frac{2u\,v}{g^2} = m\,v\,, \quad m = \frac{2u}{g^2} = \frac{Q}{2}\,,$$

$$E = \int_{-\infty}^{\infty} dx\,T^t{}_t = \frac{uv^2}{g^2} - \frac{u^3}{3g^2} - \frac{4u\mu}{g^2} = \frac{P^2}{2m} - \frac{1}{24}g^4 m^3 - \mu\,Q\,,$$

- up to a constant the dispersion relation is nonrelativistic
- the U(1) charge is twice the mass of the soliton

Undeformed NLS soliton

- The undeformed one-soliton solution exists for $\kappa < 0$

$$\kappa = -\frac{g^2}{4}\,, \quad g > 0$$

- The undeformed one-soliton solution

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$T\overline{T}$ deformed NLS soliton

- The $T\overline{T}$ deformed Lagrangian for the NLS model simplifies

$$\mathcal{L} = \frac{\frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \bar{A}\psi' - \bar{\psi}' A + \bar{A}A - U + \alpha\frac{i}{2}(\bar{A}\psi + \bar{\psi}A)(\dot{\bar{\psi}}\psi' - \bar{\psi}'\dot{\psi})}{1 - \alpha(\bar{A}A - U)}$$

$$U = -\frac{g^2}{4}(\bar{\psi}\psi)^2 - \mu\bar{\psi}\psi$$

- The deformed soliton: $\psi = \rho(x - vt)\, e^{i\phi}, \quad A = \rho_A(x - vt) e^{i\phi}$

$$\rho' = \pm\frac{2\rho\sqrt{u^2 - g^2\rho^2}}{4 + \alpha\rho^2\left(-2g^2\rho^2 + u^2 - v^2 - 4\mu\right)}, \quad \rho_A = \frac{1}{2}\rho\left(iv \pm \sqrt{u^2 - g^2\rho^2}\right)$$

$$x - vt = x_0 \pm \frac{2\coth^{-1}\left(\frac{u}{\sqrt{u^2 - g^2\rho^2}}\right)}{u} \mp \frac{\alpha\sqrt{u^2 - g^2\rho^2}\left(u^2 + 3v^2 + 12\mu + 2g^2\rho^2\right)}{6g^2}$$

$$\phi = \frac{1}{2}v(x - vt) + \frac{1}{4}t\left(u^2 + v^2 + 4\mu\right) \pm \frac{\alpha v\left(u^2 - g^2\rho^2\right)^{3/2}}{6g^2}$$

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- Set $t = 0$ and $x_0 = 0$
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- Maximum of $\rho(x)$ is u/g, and it is at $x = 0$
- ρ is a single-valued function of x only if $\rho' \neq \infty$ for all x

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Good values of parameters

- Two critical values of α

$$\alpha_- \equiv -\frac{32g^2}{(u^2 - v^2 - 4\mu)^2} < 0\,, \quad \alpha_+ \equiv \frac{4g^2}{u^2(u^2 + v^2 + 4\mu)}$$

- Good regions

A. $-\infty < u^2 - v^2 - 4\mu < 0$ and $-\infty < \alpha < \alpha_+\,, \quad \alpha_+ > 0$

B. $0 < u^2 - v^2 - 4\mu < 2u^2$ and $\alpha_- < \alpha < \alpha_+\,, \quad \alpha_+ > 0$

C. $2u^2 < u^2 - v^2 - 4\mu < 4u^2$ and $\alpha_+ < \alpha_- < \alpha < \infty$

D. $4u^2 < u^2 - v^2 - 4\mu < \infty$ and $\alpha_+ < \alpha < \infty\,, \quad \alpha_+ < \alpha_- < 0$

i. A is satisfied if $v^2 > u^2 - 4\mu$. A lower bound on v^2 if $u^2 > 4\mu$

ii. If $\mu \geq 0$ B is satisfied for all $u\,,\ v$ but C and D are not

iii. D is satisfied if $v^2 < -3u^2 - 4\mu$. An upper bound on v^2

iv. If $\mu < 0$ then all the four conditions can occur

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Shape of the deformed NLS soliton

- Q, P and E of the soliton are unchanged by the deformation
- The shape of the soliton changes
- Define its size by using the full-width-half-maximum

$$FWHM = -\alpha \frac{\sqrt{3}\, u \left(u^2 + 2v^2 + 8\mu\right)}{4g^2} + \frac{4 \log\left(2 + \sqrt{3}\right)}{u}$$

- The soliton exhibits the phenomenon of widening/narrowing the width of particles under the $T\bar{T}$ deformation
- Whether the size is increasing or decreasing depends not only on the sign of α but also on the sign of $s \equiv u^2 + 2v^2 + 8\mu$
  1. $s > 0$ for all values of u and v only if $\mu \geq 0$
  2. $s > 0$ if the soliton parameters satisfy condition A
  3. $s < 0$ for conditions C or D
  4. For parameters satisfying condition B one can have both positive and negative s if μ is negative

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Plots of the deformed NLS soliton

Set $g = 1$, $u = 1$, $v = 0$. Graphs are parametrised by μ

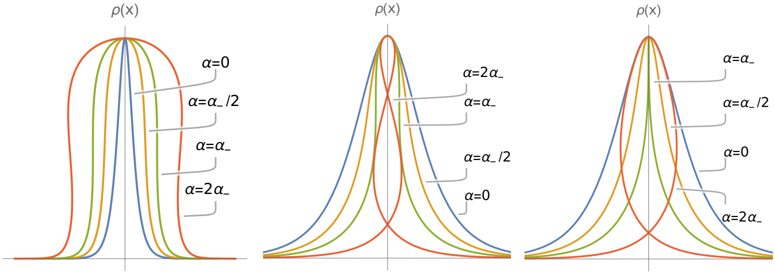

Figure: Left: Case B, $\mu = 0$, $\alpha_- = -32$, displaying formation of shockwave solution for negative α. Centre: Boundary case of B and C, $\mu = -1/4$, $\alpha_- = -8$, example of competing shockwave and narrowing behaviours creating a double-loop solution. Right: Case C, $\mu = -0.6$, $\alpha_- = -2.76817$, soliton is becoming singular at α_-, after which it forms a loop.

Plots of the deformed NLS soliton

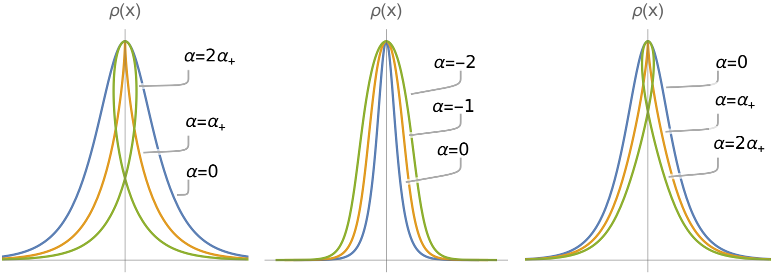

Figure: Left & Centre: Case A, $\mu = 1$, $\alpha_+ = 4/5$, displaying loop formation for $\alpha > \alpha_+ > 0$ and widening for $\alpha < 0$. Right: Case B, $\mu = 0$, $\alpha_+ = 4$, loop solution appears for $\alpha > 0$, this is the only case with a finite region of valid α.

Plots of the deformed NLS soliton

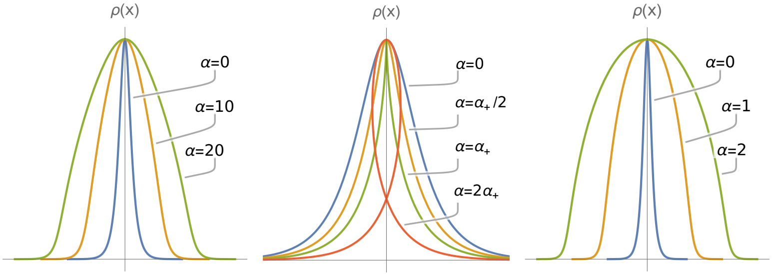

Figure: Left: Case C, $\mu = -0.6$, $\alpha_- = -2.76817$, showing regular widening solution for $\alpha > 0$. Centre & Right: Case D, $\mu = -10$, $\alpha_+ = -4/39$. Loop formation for $\alpha < \alpha_+ < 0$, widening for $\alpha > 0$. Note the varying rate of soliton widening between the two cases.

Gluing procedure

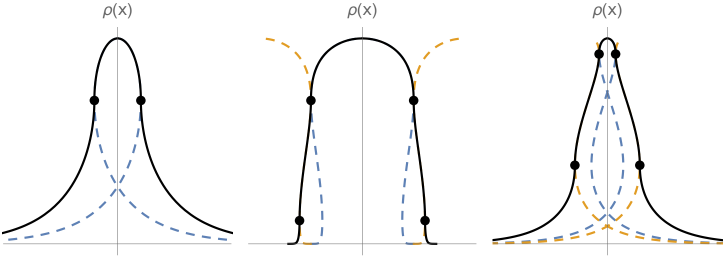

Figure: Demonstration of the gluing procedure on the loop (Left), bell (Centre) and double-loop (Right) soliton solutions, indicating the points where ρ' becomes singular.

Undeformed KdV soliton

- KdV eq is the $g = 1\,,\ h = 0$ case of Gardner eq

$$\dot{u} + \mu\, u' + 6uu' + u''' = 0$$

- We keep μ so that for $\mu < 0$ we could have left-moving solitons
- The undeformed one-soliton solution

$$u = \frac{2w^2}{\cosh^2\left(w(x - vt)\right)}\,, \qquad w = \frac{1}{2}\sqrt{v - \mu} > 0$$

$$\phi = 2w \tanh\left(w(x - vt)\right) + f(t)$$

where $f(t)$ is any function of t

- Charge Q, momentum P and energy E of the soliton

$$Q = \int_{-\infty}^{\infty} dx\, u = 4w\,, \qquad P = \int_{-\infty}^{\infty} dx\, u^2 = \frac{16}{3}w^3$$

$$E = \int_{-\infty}^{\infty} dx\,(\mu\, u^2 + 2u^3 - u'^2) = \frac{16}{3}\mu w^3 + \frac{64}{5}w^5 = \mu P + \frac{3}{5}\left(\frac{3}{2}\right)^{2/3} P^{5/3}$$

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$T\overline{T}$ deformed KdV soliton

- The $T\overline{T}$ deformed soliton depends on $f(t)$ in a nontrivial way
- Consider the simplest case $f(t) = b\,t$ where b is a constant
- Redefining ϕ as $\phi \to \phi + bt$, we find $\mathcal{L} \to \mathcal{L} - b\mathcal{J}^t$, $\mathcal{J}^t = -\partial\mathcal{L}/\partial\dot{\phi}$
- We can interpret b as the parameter of the deformation by cJ^t
- The $T\overline{T}$ deformed solution can be found by using the ansatz

$$\phi = \phi(x - v\,t) + b\,t\,,\ \mathfrak{u} = \mathfrak{u}(x - v\,t)\,,\ A = A(x - v\,t)\,,\ B = B(x - v\,t)$$

- The solution

$$\mathfrak{u}' = \pm\frac{\mathfrak{u}\sqrt{4\tilde{w}^2 - 2\mathfrak{u}}}{1 + \alpha\mathfrak{u}^2\,(4\mathfrak{u} - 8\tilde{w}^2 - \alpha b^2)}\,, \quad \phi' = \frac{\mathfrak{u} - \alpha\,b\,\mathfrak{u}^2}{1 + \alpha\mathfrak{u}^2\,(4\mathfrak{u} - 8\tilde{w}^2 - \alpha b^2)}$$

$$B = \left(\mu + 4\tilde{w}^2\right)\mathfrak{u} + b\,,\quad A = \pm\mathfrak{u}\sqrt{4\tilde{w}^2 - 2\mathfrak{u}}\,,\quad \tilde{w}^2 = \frac{v - \mu - \alpha\,b^2}{4} = w^2 - \frac{\alpha\,b^2}{4}$$

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$T\overline{T}$ deformed KdV soliton

- Deformed energy and momentum

$$E = \frac{16}{15}\tilde{w}^3\left(12\tilde{w}^2 + 5(\mu - \alpha b^2)\right)\;, \quad P = \frac{16}{3}\tilde{w}^3$$

$$E(P) = P\left(\mu - \alpha b^2\right) + \frac{3}{5}\left(\frac{3}{2}\right)^{2/3} P^{5/3}$$

- The previously identical conserved charges of $\mathcal{J}^t$ and ϕ' become independent

$$Q = \int dx \mathcal{J}^t = 4\tilde{w}\left(1 + \frac{4}{3}\tilde{w}^2\alpha b\right)\;, \quad Q_\phi = \int dx \phi' = 4\tilde{w}\left(1 - \frac{4}{3}\tilde{w}^2\alpha b\right)$$

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$T\bar{T}$ deformed KdV soliton

- Integrating the equation for $\mathfrak{u}'$, we find

$$x - vt = x_0 \pm \frac{\operatorname{arctanh}\left(\frac{\sqrt{4\tilde{w}^2 - 2\mathfrak{u}}}{2\tilde{w}}\right)}{\tilde{w}} \mp \frac{1}{15}\sqrt{2}\alpha\sqrt{2\tilde{w}^2 - \mathfrak{u}}\left(4\left(2\tilde{w}^2 - \mathfrak{u}\right)\left(3\mathfrak{u} + 4\tilde{w}^2\right) + 5\alpha b^2\left(\mathfrak{u} + 4\tilde{w}^2\right)\right)$$

- It displays both shockwave and looping solutions
- The full-width half-maximum of the soliton

$$FWHM = \frac{2\operatorname{arcoth}\left(\sqrt{2}\right)}{\tilde{w}} - \frac{2\sqrt{2}}{15}\alpha\tilde{w}^3\left(25\alpha b^2 + 28\tilde{w}^2\right)$$

- For positive α it decreases
- For negative α it may increase or decreases depending on b

$T\overline{T}$ deformed KdV soliton

- Integrating the equation for u', we find

$$x - vt = x_0 \pm \frac{\operatorname{arctanh}\left(\frac{\sqrt{4\tilde{w}^2 - 2u}}{2\tilde{w}}\right)}{\tilde{w}} \mp \frac{1}{15}\sqrt{2}\alpha\sqrt{2\tilde{w}^2 - u}\left(4\left(2\tilde{w}^2 - u\right)\left(3u + 4\tilde{w}^2\right) + 5\alpha b^2\left(u + 4\tilde{w}^2\right)\right)$$

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- Critical values of α

$$\alpha_- = -\frac{\sqrt{4w^4 + 2|b|} - 2w^2}{b^2}\,, \quad \alpha_+^{(2)} = \frac{2w^2 - \sqrt{4w^4 - 2|b|}}{b^2}\,,$$

$$\alpha_+^{(3)} = \frac{2w^2 + \sqrt{4w^4 - 2|b|}}{b^2}$$

- $\alpha_+^{(1)}$ is the positive root smaller than $\frac{2w^2}{b^2}$ of the equation

$$1 - \frac{128}{27}\alpha\left(w^2 - \frac{\alpha b^2}{8}\right)^3 = 0$$

$T\overline{T}$ deformed KdV soliton

Bad regions

$$\text{Loop:}\ \begin{cases} b \neq 0\,, & \alpha < \alpha_- < 0 \\ b \neq 0\,,\ 4w^8 > b^2\,, & \alpha_+^{(2)} < \alpha < \alpha_+^{(3)} < \frac{4w^2}{b^2} \end{cases}$$

$$\text{Bell or Double Loop:}\ \begin{cases} b = 0\,, & \alpha > \frac{27}{128w^6} \\ b \neq 0\,,\ 4w^8 > b^2\,, & 0 < \alpha_+^{(1)} < \alpha < \alpha_+^{(2)} \end{cases}$$

Plots of the deformed KdV soliton

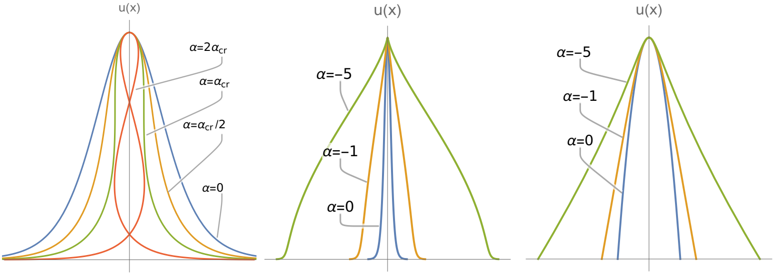

Figure: KdV soliton solutions for $w = 1,\ b = 0$. Double-loop solution forms only for $\alpha > \alpha_c = 27/128$. For $\alpha < 0$, solution remains single-valued and increases in width. Rightmost plot examines peak of $\alpha < 0$ plot, indicating that the solution remains smooth at x=0.

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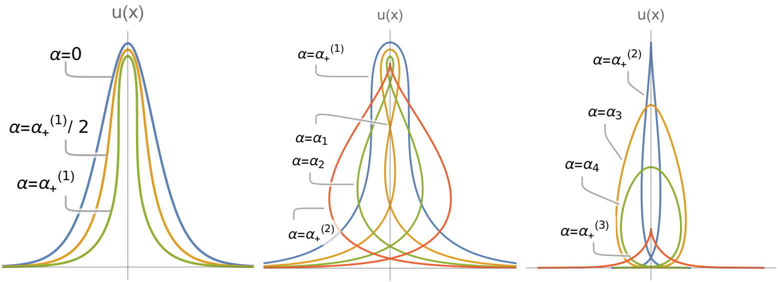

Figure: KdV soliton for $w = 1$, $b = 1, \alpha > 0$, transitioning between different types of multi-valued solutions. Left: Width is decreasing with increasing α, $\alpha_+^{(1)} \approx 0.23$. Centre: Formation of double-loop solution for $\alpha_+^{(1)} < \alpha < \alpha_+^{(2)}$, with a singular solution at $\alpha = \alpha_+^{(2)} \approx 0.59$. The intermediate values are equally spaced, $\alpha_1 = \frac{2\alpha_+^{(1)}+\alpha_+^{(2)}}{3}$, $\alpha_2 = \frac{\alpha_+^{(1)}+2\alpha_+^{(2)}}{3}$. Right: Amplitude decreasing, transitioning to singular peak at $\alpha = \alpha_+^{(3)} \approx 3.41$. $\alpha_3 = \frac{2\alpha_+^{(2)}+\alpha_+^{(3)}}{3}$, $\alpha_4 = \frac{\alpha_+^{(2)}+2\alpha_+^{(3)}}{3}$.

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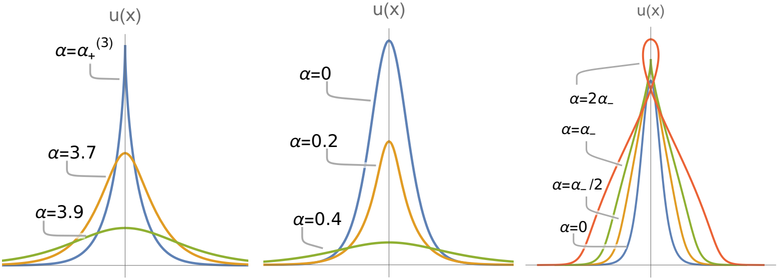

Figure: Left: Continuation of evolution from figure 6, displaying single-valued solution for $\alpha > \alpha_+^{(3)} \approx 3.41$. The extreme flattening of the solution in the limit $\alpha \to 4$ is due to $\tilde{w} \to 0$. Centre: With $w = 1,\ b = 3$, the soliton remains regular for all $0 < \alpha < 4/9$, after which it ceases to exist in a similar fashion. Right: $w = 1,\ b = 1,\ \alpha < 0,\ \alpha_- \approx -4.4$. Solution widens, but with nonzero b develops into a loop solution.

Comments II

- Soliton's width depends on α confirming the general phenomenon of widening/narrowing the width of particles under the $T\overline{T}$ deformation Cardy, Doyon '20
- Whether soliton's size is increasing or decreasing depends on the sign of α, and on the potential and soliton parameters
- In the NLS case this is caused by the addition of the time component of the conserved U(1) current to the seed model
- After the $T\overline{T}$ deformation this cannot be undone by a time dependent U(1) transformation, and leads to substantial changes in the soliton's properties
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- In the absence of the chemical potential the width is increasing for $\alpha < 0$ and decreasing for $\alpha > 0$ which is opposite to what was observed in Cardy, Doyon '20 & Jiang '20
- It is because the energy of the NLS soliton is given by $E = \frac{P^2}{2m} - \frac{1}{24} g^4 m^3 - \mu\, Q$, and for $\mu = 0$ its rest energy is negative
- The existence of the rest energy means that the $T\overline{T}$ deformation is a mixture of the $T\overline{T}$ deformation with a stress-energy tensor shifted so that the rest energy is zero, and the JP deformation discussed in Cardy, Doyon '20 & Jiang '20
- If the chemical potential is sufficiently negative then the width is widening or narrowing in accord with Cardy, Doyon '20
- In the KdV case with the parameter $b = 0$ the width of the deformed soliton again behaves oppositely to Cardy, Doyon '20 & Jiang '20
- Since the rest energy of the soliton is zero, one may conclude that the effect of "pure" $T\overline{T}$ deformation is in fact opposite to what was observed for the JP deformation Cardy, Doyon '20 & Jiang '20

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- Another common property of the deformed solitons is that for any values of the parameters there is at least one critical value $\alpha_{\rm cr}$ at which solitons begin to exhibit the shock-wave behaviour
- We proposed that for values of α beyond $\alpha_{\rm cr}$ a soliton solution may be constructed by gluing together the two branches of the soliton solution at the points where the first derivative of the soliton field diverges
- Despite the divergency, the soliton energy and momentum are finite, and the dispersion relation is defined for all values of α
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Dzhordzhadze, Pogrebkov, Polivanov '1979

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- The $T\overline{T}$ deformed KdV equation admits at least a one-parameter family of one-soliton solutions
- The extra parameter b can be introduced explicitly in the $T\overline{T}$ deformed Lagrangian by shifting the field ϕ by bt, and requiring that ϕ asymptotes to constants at space infinities
- Then, b can be interpreted as the parameter of the deformation by $\mathcal{J}^t$ of the conserved current due to the invariance of the $T\overline{T}$ deformed Gardner model under constant shifts of ϕ
- Since b modifies soliton's properties, e.g. it appears in the dispersion relation, it is probably the right interpretation
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Open questions

- Put the models on a circle and look for all possible solutions including those with energy divergent in the limit $\alpha \to 0$
- Find Lax pairs for the deformed models. Lax pairs of several models including the NLS model were found in Chen, Hou, Tian '21
- Apply the method to the matrix NLS model and the LL model
- Generalise their method to models of the Gardner type where auxiliary fields cannot be eliminated
- Understanding the Poisson structure and developing a Hamiltonian formulation is important and probably very hard
- Given a Lax pair (V, U) and a Hamiltonian formulation of the NLS model, one can calculate the Poisson bracket between U's, and see how the r-matrix structure is modified, and whether it can be quantised

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