Deforming the ODE/IM correspondence with $T\bar{T}$

Istituto Nazionale di Fisica Nucleare
Sezione di Torino

APCTP Focus
Exact results on irrelevant deformations of QFT
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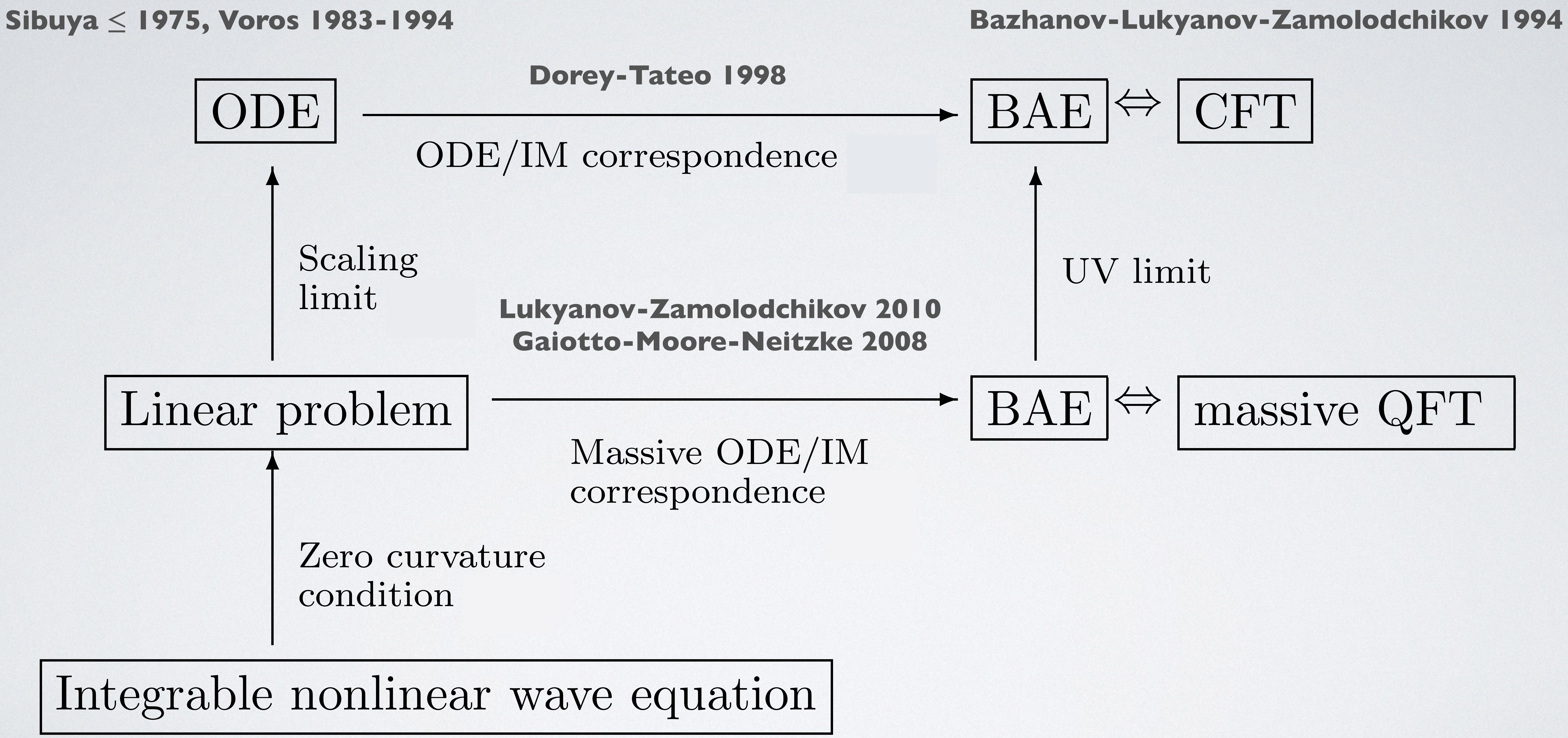
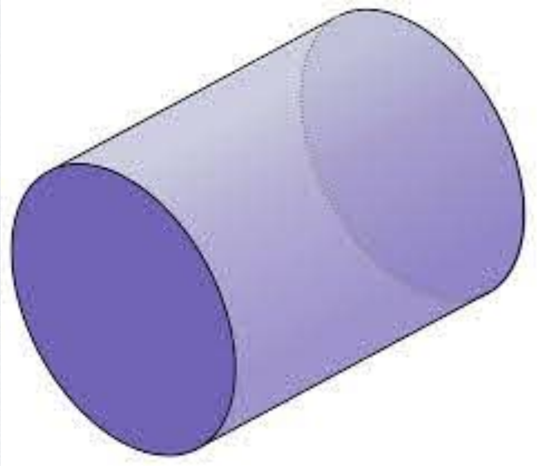
Università degli Studi di Torino

Roberto Tateo

Current collaborators

Fabrizio Aramini, Nicolò Brizio, Stefano Negro


Sibuya ≤ 1975, Voros 1983-1994
Bazhanov-Lukyanov-Zamolodchikov 1994
Dorey-Tateo 1998
ODE
BAE
⇔
CFT
ODE/IM correspondence
Scaling limit
UV limit
Lukyanov-Zamolodchikov 2010
Gaiotto-Moore-Neitzke 2008
Linear problem
BAE
⇔
massive QFT
Massive ODE/IM correspondence
Zero curvature condition
Integrable nonlinear wave equation



The sin-Gordon NLIE: IM-side

$$\epsilon(\theta) = r\sinh(\theta) + 2\pi k - 2\int_{\mathbb{R}} d\theta' \mathcal{K}(\theta-\theta')\,\Im m \ln\left(1+e^{-i\epsilon(\theta-i0)}\right)$$

(This is for the vacuum states, more complicated integration contours appear for excited states.)

$$\mathcal{K}(\theta) = \frac{1}{i2\pi}\frac{d}{d\theta}\ln S_{sG}(\theta) \qquad r = mR = s^{1+\alpha}\frac{2\sqrt{\pi}\Gamma(\frac{1}{2\alpha})}{\Gamma(\frac{1}{2}+\frac{1}{2\alpha})} \qquad \alpha = \beta^{-2}-1 > 1$$

where m is the soliton mass, R the circumference of the cylinder and s is the scale parameter, useful for later considerations in the ODE/IM setup

Starting from the NLIE, and setting:

$$-i\epsilon(\theta) = \ln\left(\frac{Q(\theta+i\pi/\alpha)}{Q(\theta-i\pi/\alpha)}\right)$$

where Q is the field-theory version of the Baxter's Q-function: $T(\theta)Q(\theta) = Q(\theta+i\pi/\alpha) + Q(\theta-i\pi/\alpha)$

one can easily extract the large-θ expansion for Q

$$\ln Q\left(\theta+i\frac{\pi(\alpha+1)}{2\alpha},k\right)\sim\frac{re^{\theta}}{4\cos(\frac{\pi}{2\alpha})}+i\pi k+\frac{1}{2}\ln\mathscr{G}-\sum_{n=1}^{\infty}J_{2n-1}e^{-(2n-1)\theta}+\sum_{n=1}^{\infty}G_n e^{-2\alpha n\theta},\quad(\theta\to\infty)$$

$$\ln Q\left(\theta+i\frac{\pi(\alpha+1)}{2\alpha},k\right)\sim\frac{re^{-\theta}}{4\cos(\frac{\pi}{2\alpha})}+i\pi k-\frac{1}{2}\ln\mathscr{G}-\sum_{n=1}^{\infty}\bar{J}_{2n-1}e^{(2n-1)\theta}+\sum_{n=1}^{\infty}\bar{G}_n e^{2\alpha n\theta},\quad(\theta\to-\infty)$$

The coefficients J_{2n-1} , $\bar{J}_{2n-1}$, G_n , $\bar{G}_n$ are the local and non-local integrals of motion, respectively.

$$J_{2n-1}=-\frac{r\delta_{n,1}}{4\cos(\frac{\pi}{2\alpha})}+\frac{(-1)^{n+1}}{\sin\left(\frac{\pi(2n-1)}{2\alpha}\right)}\int_{\mathbb{R}}\frac{d\theta}{\pi}\Im m\, e^{(2n-1)(\theta-i0)}\ln\left(1+e^{-i\epsilon(\theta-i0)}\right)$$

$$\bar{J}_{2n-1}=-\frac{r\delta_{n,1}}{4\cos(\frac{\pi}{2\alpha})}-\frac{(-1)^{n+1}}{\sin\left(\frac{\pi(2n-1)}{2\alpha}\right)}\int_{\mathbb{R}}\frac{d\theta}{\pi}\Im m\, e^{-(2n-1)(\theta-i0)}\ln\left(1+e^{-i\epsilon(\theta-i0)}\right)$$

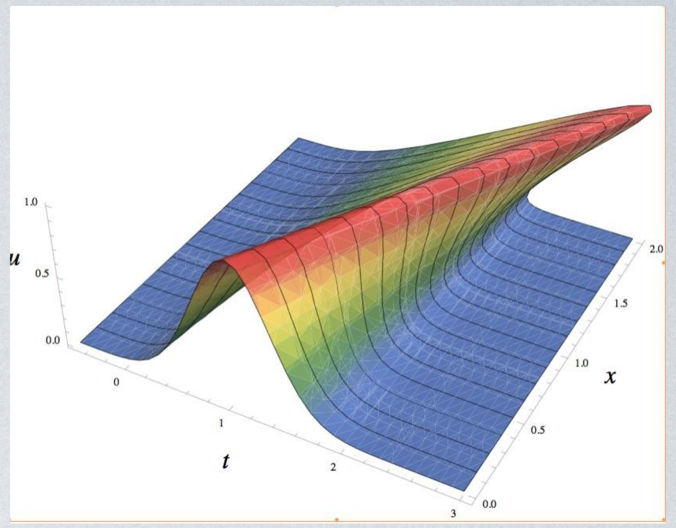
In terms of J_1 and $\bar{J}_1$ the total energy and momentum are

$$E = \frac{m}{2}\sin\left(\frac{\pi}{2\alpha}\right)\left(J_1 + \bar{J}_1 + \frac{r}{2\cos(\frac{\pi}{2\alpha})}\right), \qquad P = \frac{m}{2}\sin\left(\frac{\pi}{2\alpha}\right)\left(J_1 - \bar{J}_1\right)$$

where

$$2m\sin\left(\frac{\pi}{2\alpha}\right) = m_B$$

for $\alpha > 1$, is the mass of the fundamental breather.

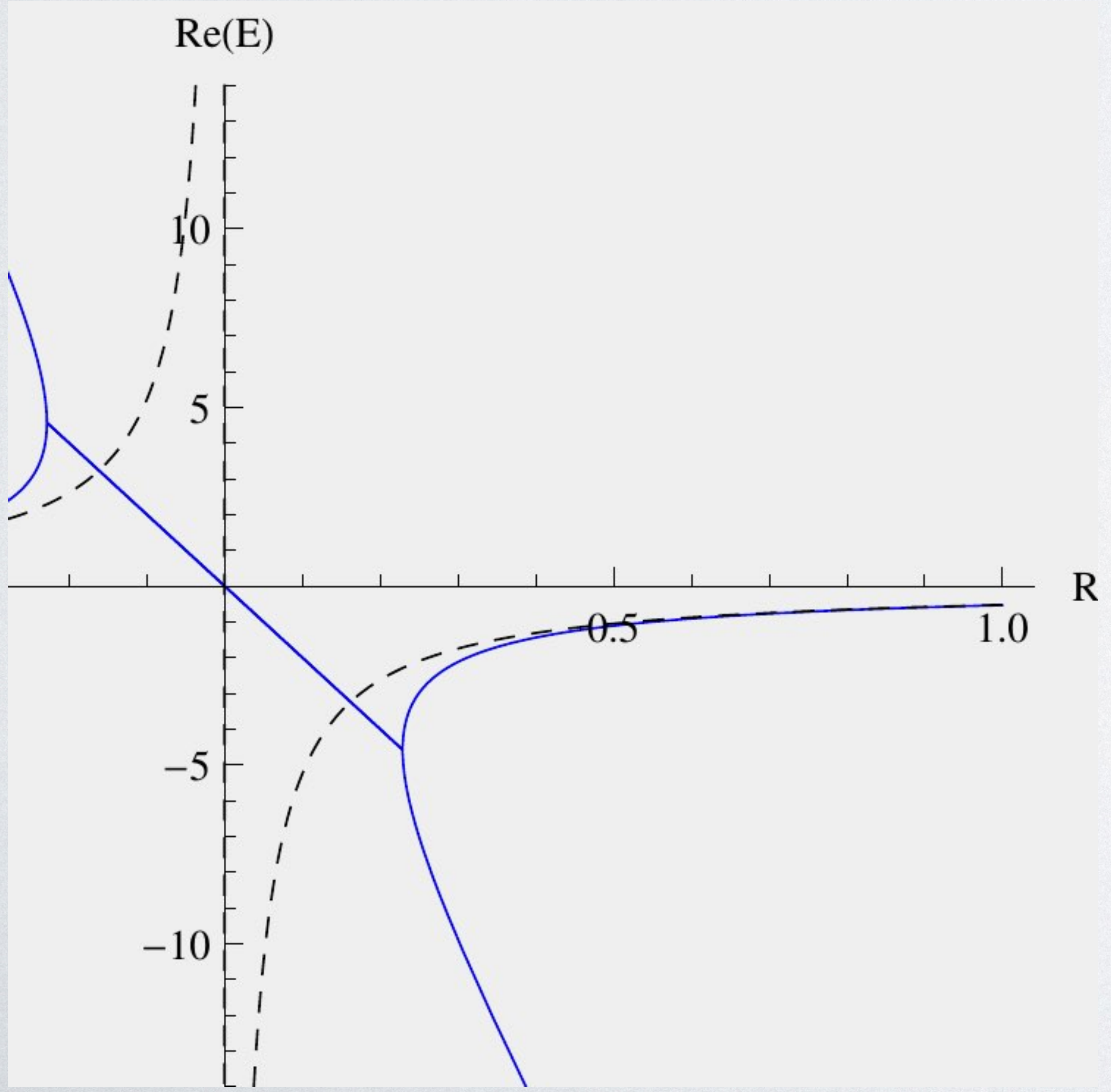


Including the $T\bar{T}$ deformation

$$\mathscr{K}(\theta) \to \mathscr{K}(\theta) - \tau \frac{m^2}{2\pi} \cosh(\theta) \iff mR \sinh(\theta) \to m\mathscr{R}_0 \sinh(\theta - \theta_0)$$

where $\mathscr{R}_0$ and θ_0 are defined through the following relations

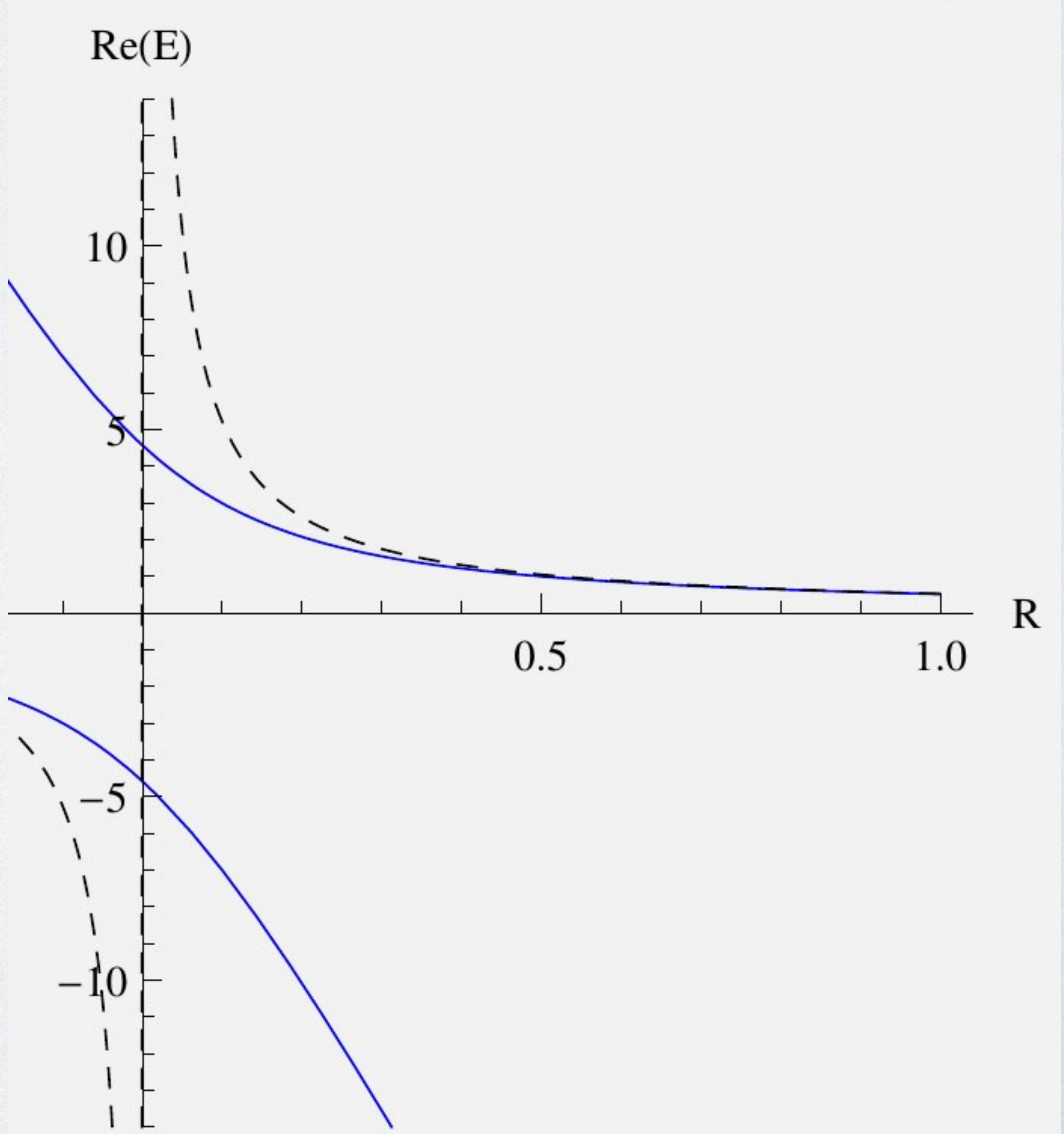
$$\mathscr{R}_0 \cosh\left(\theta_0\right) = R + \tau\, E(R,\tau)\ , \quad \mathscr{R}_0 \sinh\left(\theta_0\right) = \tau\, P(R,\tau)$$



$$\partial_\tau E(R,\tau) = E(R,\tau)\partial_R E(R,\tau) + \frac{P^2(R)}{R}$$

(The Burgers-type equation)

$$P = 0 \to E(R,\tau) = E(R + \tau E(R,\tau), 0)$$



Classical (modified) sinh-Gordon model: ODE-side

$$\partial_z \partial_{\bar{z}} \eta - e^{2\eta} + p(z)\, p(\bar{z}) e^{-2\eta} = 0$$

(Modified sinh-Gordon equation)

with the functions $p(z)$ of the form

$$p(z) = z^{2\alpha} - s^{2\alpha}$$

Modified sinh-Gordon and standard sinh-Gordon equations are related by a simple change of variables

$$dw = \sqrt{p(z)}dz, \quad d\bar{w} = \sqrt{p(\bar{z})}d\bar{z}$$

$$\hat{\eta} = \eta - \frac{1}{4} \ln p\bar{p}$$

$$\partial_w \partial_{\bar{w}} \hat{\eta} - e^{2\hat{\eta}} + e^{-2\hat{\eta}} = 0$$

(sinh-Gordon equation)

Lax pair

$$\boldsymbol{D} = \partial_z + \frac{1}{2}\partial_z\eta\sigma^3 - e^{\theta}\left[\sigma^+\ e^{\eta} + \sigma^- p(z)\, e^{-\eta}\right]\ ,\ \bar{\boldsymbol{D}} = \partial_{\bar z} - \frac{1}{2}\partial_{\bar z}\eta\ \sigma^3 - e^{-\theta}\left[\sigma^-\ e^{\eta} + \sigma^+ p(\bar z)\, e^{-\eta}\right]$$

$$\boldsymbol{D}\boldsymbol{\Psi} = 0\,, \qquad \bar{\boldsymbol{D}}\boldsymbol{\Psi} = 0$$

Lax equations (associate linear system)

with the parametrisation

$$\boldsymbol{\Psi} = \begin{pmatrix} e^{\frac{\theta}{2}}\ e^{\frac{\eta}{2}}\ \psi \\ e^{-\frac{\eta}{2}}\ e^{-\frac{\theta}{2}}\,(\partial_z + \partial_z\eta)\,\psi \end{pmatrix} = \begin{pmatrix} e^{-\frac{\eta}{2}}\ e^{\frac{\theta}{2}}\,(\partial_{\bar z} + \partial_{\bar z}\eta)\,\bar\psi \\ e^{\frac{\eta}{2}}\ e^{-\frac{\theta}{2}}\ \bar\psi \end{pmatrix}$$

$$\left[\partial_z^2 - u(z,\bar z) - e^{2\theta}\ \ p(z)\right]\psi = 0\ , \quad \left[\partial_{\bar z}^2 - \bar u(z,\bar z) - e^{-2\theta}\,p(\bar z)\right]\bar\psi = 0$$

and

$$u(z,\bar z) = (\partial_z\eta)^2 - \partial_z^2\eta\ , \qquad \bar u(z,\bar z) = (\partial_{\bar z}\eta)^2 - \partial_{\bar z}^2\eta$$

Consider first the light-cone limit $\bar{z} \to 0$ in which η has the asymptotic

$$\eta \to l\,\log(z\bar{z}) + \eta_0 + \ldots \qquad (l = 2|k| - 1/2)$$

For the ODE/IM : $\eta(|z|, \phi + \frac{\pi}{\alpha}) = \eta(|z|, \phi), \quad \eta(|z|, \phi) \sim \alpha \ln |z|$

In the limit $z \sim s \to 0, \theta \to \infty$, with the combinations

$$x = e^{\frac{\theta}{1+\alpha}}\, z\,, \qquad E = s^{2\alpha}\, e^{\frac{2\theta\alpha}{1+\alpha}}\,,$$

one finds the Schroedinger equation for the anharmonic oscillator (original ODE/IM):

$$\left[-\partial_x^2 + \frac{l(l+1)}{x^2} + x^{2\alpha} \right] \psi = E\,\psi$$

The classical local integrals of motion are

$$J_{2n-1} = \frac{1}{2(2n-1)\sin(\frac{\pi(2n-1)}{2\alpha})} \int_{\gamma_1} dw \hat{P}_{2n} + d\bar{w}\hat{R}_{2n-2}, \quad \bar{J}_{2n-1} = \frac{1}{2(2n-1)\sin(\frac{\pi(2n-1)}{2\alpha})} \int_{\bar{\gamma}_1} d\bar{w}\hat{\bar{P}}_{2n} + dw\hat{\bar{R}}_{2n-2}$$

$$\partial_{\bar{w}}\hat{P}_{2n} = \partial_w \hat{R}_{2n-2}, \qquad \partial_w \hat{\bar{P}}_{2n} = \partial_{\bar{w}}\hat{\bar{R}}_{2n-2}$$

$$T_2(\mathbf{w}) = \frac{1}{2}\hat{u}(w,\bar{w}) = \hat{P}_2, \quad \bar{T}_2(\mathbf{w}) = \frac{1}{2}\hat{\bar{u}}(w,\bar{w}) = \hat{\bar{P}}_2,$$

$$\Theta_0(\mathbf{w}) = \bar{\Theta}_0(\mathbf{w}) = \hat{R}_0 + 1 = \hat{\bar{R}}_0 + 1 = e^{-2\hat{\eta}}$$

$$\hat{u}(w,\bar{w}) = (\partial_w \hat{\eta})^2 - \partial_w^2 \hat{\eta} \quad , \quad \hat{\bar{u}}(w,\bar{w}) = (\partial_{\bar{w}}\hat{\eta})^2 - \partial_{\bar{w}}^2 \hat{\eta}$$

$$E = \mathscr{E} + \bar{\mathscr{E}}\,,\; P = \mathscr{E} - \bar{\mathscr{E}}$$

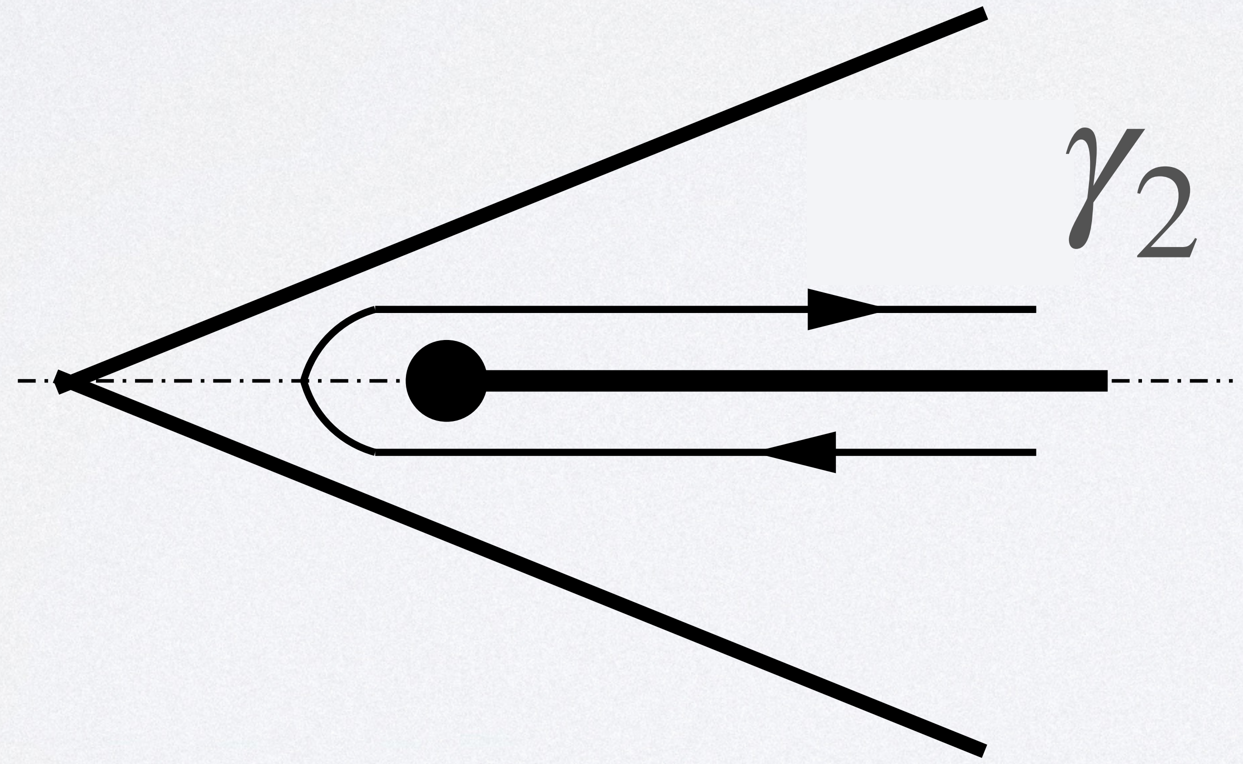
$$\mathscr{E} = \frac{m}{4}\int_{\gamma_1} dw\, T_2(\mathbf{w}) + d\bar{w}\,\Theta_0(\mathbf{w})\,, \quad \bar{\mathscr{E}} = \frac{m}{4}\int_{\bar{\gamma}_1} dw\,\bar{\Theta}_0(\mathbf{w}) + d\bar{w}\,\bar{T}_2(\mathbf{w})$$

The ODE/IM correspondence claim is:

$$\{J_{2n-1}(R), \bar{J}_{2n-1}(R)\}_{Quantum} = \{J_{2n-1}(R), \bar{J}_{2n-1}(R)\}_{Classical}$$

The circumference R

$$R = \frac{1}{m\tan(\frac{\pi}{2\alpha})}\left(\int_{\gamma_1} d\bar{w} + \int_{\bar{\gamma}_1} dw\right) = \frac{1}{m\tan(\frac{\pi}{2\alpha})}\left(\int_{\gamma_2} \bar{p}^{1/2} d\bar{z} + \int_{\bar{\gamma}_2} p^{1/2} dz\right)$$



Classical $T\bar{T}$ as a dynamical change of coordinates

$$J = \frac{1}{\Delta(\mathbf{w})} \begin{pmatrix} 1 + 2t\Theta_0(\mathbf{w}) & -2tT_2(\mathbf{w}) \\ -2t\bar{T}_2(\mathbf{w}) & 1 + 2t\bar{\Theta}_0(\mathbf{w}) \end{pmatrix}$$

$$\begin{pmatrix} dw \\ d\bar{w} \end{pmatrix} = J^T \begin{pmatrix} dx \\ d\bar{x} \end{pmatrix}$$

$$\Theta_0 = \bar{\Theta}_0$$

$$\tau = -\frac{8}{m^2 \tan\left(\frac{\pi}{2\alpha}\right)} t$$

Now we wish to compute what we expect to be the $T\bar{T}$ perturbed circumference R of the theory, defined as

$$\tilde{R}(R,t) = \frac{1}{m \tan\left(\frac{\pi}{2\alpha}\right)} \left(\int_{\gamma_t} d\bar{w} + \int_{\bar{\gamma}_t} dw \right)$$

$$\int_{\bar{\gamma}_t} dw = \int_{\bar{\gamma}_1} \frac{(1 + 2t\Theta_0(\mathbf{w}((\mathbf{x})))}{\Delta(\mathbf{w}(\mathbf{x}))} dx - 2t \frac{\bar{T}_2(\mathbf{w}(\mathbf{x}))}{\Delta(\mathbf{w}(\mathbf{x}))} d\bar{x} = \int_{\bar{\gamma}_1} dx - 2t \int_{\bar{\gamma}_1} \bar{\Theta}_0(\mathbf{x}) dx + \bar{T}_2(\mathbf{x}) d\bar{x}$$

where

$$\Delta(\mathbf{w}) = (1 + 2t\Theta_0(\mathbf{w}))(1 + 2t\bar{\Theta}_0(\mathbf{w})) - 4t^2 T_2(\mathbf{w}) \bar{T}_2(\mathbf{w})$$

Therefore

$$\tilde{R}(R, t) = R - \frac{8}{m^2 \tan\left(\frac{\pi}{2\alpha}\right)} t E(R, t) = R + \tau E(R, \tau)$$

We would like to prove the validity of the Burgers equation on this classical (ODE) side of the correspondence

$$\frac{dE(R,\tau)}{d\tau} = \frac{dE(R,\tau)}{dR} E(R,\tau)$$

We consider the curve $\tilde{R}(R,\tau)$ = constant, with $R = R(\tau)$, and take the derivative with respect to τ :

$$0 = \frac{dR}{d\tau} + E(R,\tau) + \tau \frac{dE(R,\tau)}{d\tau} + \tau \frac{dE(R,\tau)}{dR} \frac{dR}{d\tau}$$

$E(R,\tau)$ fulfils the Burgers equation if

$$\frac{dR}{d\tau} = - E(R,\tau)$$

To show this, it is necessary to clarify what $\tilde{R}(R,\tau) = constant$ means in this context.

$$\tilde{R}(R,\tau) = \frac{1}{m\tan(\frac{\pi}{2\alpha})}\left(\int_{\gamma_t} d\bar{w} + \int_{\bar{\gamma}_t} dw\right)$$

Clearly, $\tilde{R}$ is completely determined by the integration contour γ_t , so keeping $\tilde{R}$ constant means keeping γ_t fixed.

At γ_t fixed, we need to find how γ_1 changes under an infinitesimal variation of t:

$$R(t) = \frac{1}{m\tan\left(\frac{\pi}{2\alpha}\right)}\left(\int_{\gamma_{1,x}} d\bar{x} + \int_{\bar{\gamma}_{1,x}} dx\right)$$

$$R(t-\delta t) = \frac{1}{m\tan\left(\frac{\pi}{2\alpha}\right)}\left(\int_{\gamma_{1,y}} d\bar{y} + \int_{\bar{\gamma}_{1,y}} dy\right)$$

The map between $\boldsymbol{x}$ and $\boldsymbol{y}$ coordinates is

$$\begin{pmatrix} dy \\ d\bar{y} \end{pmatrix} = (J^T)^{-1}(t-\delta t)J^T(t)\begin{pmatrix} dx \\ d\bar{x} \end{pmatrix}$$

Explicitly:

$$dy = dx - 2\delta t(\bar{\Theta}_0(\mathbf{x})dx + \bar{T}_2(\mathbf{x})d\bar{x})$$
$$d\bar{y} = d\bar{x} - 2\delta t(T_2(\mathbf{x})dx + \Theta_0(\mathbf{x})d\bar{x})$$

giving

$$\begin{aligned} R(t-\delta t) &= \frac{1}{m\tan\left(\frac{\pi}{2\alpha}\right)}\left(\int_{\bar{\gamma}_{1,y}} dy + \int_{\gamma_{1,y}} d\bar{y}\right) \\ &= \frac{1}{m\tan\left(\frac{\pi}{2\alpha}\right)}\left(\int_{\bar{\gamma}_{1,x}} dx + \int_{\gamma_{1,x}} d\bar{x} - 2\delta t\left(\int_{\bar{\gamma}_{1,x}} (\bar{\Theta}_0(\mathbf{x})dx + \bar{T}_2(\mathbf{x})d\bar{x}) + \int_{\gamma_{1,x}} (T_2(\mathbf{x})dx + \Theta_0(\mathbf{x})d\bar{x})\right)\right) \\ &= R(t) - \frac{8\delta t}{m^2\tan\left(\frac{\pi}{2\alpha}\right)}E(R,t) \end{aligned}$$

Therefore

$$\frac{dR}{d\tau} = -E(R,\tau)$$

In conclusion we have

$$\frac{dE(R,\tau)}{d\tau} = \frac{dE(R,\tau)}{dR} E(R,\tau)$$

and

$$E(R,\tau) = E(\tilde{R}(R,\tau),0)$$

on both Classical and Quantum sides of the correspondence.

$\Longrightarrow$ **If the ODE/IM correspondence is valid at $\tau = 0$ it is also valid at $\tau \neq 0$!**

Conclusions

1) The ODE/IM correspondence is valid also in $T\bar{T}$ deformed integrable models.

2) The analysis can be done for generic polynomial potential and for a wide class of models (the outcome is the same)

3) The interpretation of this result in the Amplitudes/Wilson loop AdS/CFT framework is still missing.

Thank you for your attention !!