A Deformed, Integrable, Non-Diagonalizable Spin Chain from QFT

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Based on

- Earlier work with Asger Ipsen and Leo Zippelius:
  The One-Loop Spectral Problem of Strongly Twisted $\mathcal{N} = 4$ Super Yang-Mills Theory, `arXiv:1812.08794`.

- Recent work with Changrim Ahn:
  The Integrable (Hyper)eclectic Spin Chain, `arXiv:2010.14515`.

- Work in preparation with Changrim Ahn and Luke Corcoran.

Motivations and Disclaimer

- There has been some recent interest in a certain *deformation* of planar $\mathcal{N}=4$ Super Yang-Mills Theory: The strong twisting limit.

- The deformation is *very relevant*: It leads to a non-unitary logarithmic conformal quantum field theory, while preserving integrability.

- We looked into this in the simplest possible setting: The one-loop dilatation operator. We found that curious new challenges arise for the integrability program.

- In fact, some novel, non-diagonalizable, but still *exactly solvable* integrable spin chains arise, encoding fascinating combinatorial problems.

Strongly Twisted $\mathcal{N}$=4 Super Yang-Mills Theory, I

Start from planar, integrable, three-parameter γ-deformed $\mathcal{N}$=4 SYM.

Perform double-scaling limit: [O. Gürdoğan, V. Kazakov '15; Sieg, Wilhelm '16; Kazakov et.al. '18].

$$g = \frac{\sqrt{\lambda}}{4\pi} \longrightarrow 0 \quad \text{and} \quad q_j = e^{-i\gamma_j/2} \longrightarrow \infty \quad \text{or} \quad q_j = e^{-i\gamma_j/2} \longrightarrow 0$$

such that for each $j = 1, 2, 3$ either $g\, q_j$ or else $g\, q_j^{-1}$ is held fixed.

This yields $2^3 = 8$ different strong twisting limits: Write $q_j := \varepsilon^{\mp 1}\, \xi_j^{\pm}$, replace $g \to \varepsilon\, g$, and take ε to zero. For $(q_1, q_2, q_3) = (\infty, \infty, \infty)$:

$$\begin{aligned}\mathcal{L}_{\text{int}} \;=\; & -g^2\, N\, \mathrm{Tr}\left((\xi_3^+)^2\, \phi_1^\dagger \phi_2^\dagger \phi^1 \phi^2 + (\xi_2^+)^2\, \phi_3^\dagger \phi_1^\dagger \phi^3 \phi^1 + (\xi_1^+)^2\, \phi_2^\dagger \phi_3^\dagger \phi^2 \phi^3 \right) \\ & -g\, N\, \mathrm{Tr}\left(i\, \sqrt{\xi_2^+ \xi_3^+} (\psi^3 \phi^1 \psi^2 + \bar{\psi}_3 \phi_1^\dagger \bar{\psi}_2) + \text{cyclic} \right)\end{aligned}$$

Gauge fields "decouple".

Strongly Twisted $\mathcal{N}$=4 Super Yang-Mills Theory, II

Look at the other 7 cases. For $(q_1, q_2, q_3) = (0,0,0)$ one has the equivalent

$$\mathcal{L}_{\text{int}} = N\,\mathrm{Tr}\left((\xi_3^-)^{-2}\,\phi_2^\dagger\phi_1^\dagger\phi^2\phi^1 + (\xi_2^-)^{-2}\,\phi_1^\dagger\phi_3^\dagger\phi^1\phi^3 + (\xi_1^-)^{-2}\,\phi_3^\dagger\phi_2^\dagger\phi^3\phi^2\right) + \mathrm{Tr}\left(i(\xi_2^-\xi_3^-)^{-\frac{1}{2}}(\psi^2\phi^1\psi^3 + \bar{\psi}_2\phi_1^\dagger\bar{\psi}_3) + \text{cyclic}\right)$$

The other six limits are different, but once again equivalent to each other. For example, for $(q_1, q_2, q_3) = (\infty, \infty, 0)$ we have

$$\mathcal{L}_{\text{int}} = N\,\mathrm{Tr}\Bigg((\xi_3^-)^{-2}\,\phi_2^\dagger\phi_1^\dagger\phi^2\phi^1 + (\xi_2^+)^2\,\phi_3^\dagger\phi_1^\dagger\phi^3\phi^1 + (\xi_1^+)^2\,\phi_2^\dagger\phi_3^\dagger\phi^2\phi^3 + \sqrt{\frac{\xi_2^+}{\xi_3^-}}\left(\bar{\psi}_1\phi^1\bar{\psi}_4 - \psi^1\phi_1^\dagger\psi^4\right) - \sqrt{\frac{\xi_1^+}{\xi_3^-}}\left(\bar{\psi}_4\phi^2\bar{\psi}_2 - \psi^4\phi_2^\dagger\psi^2\right) - i\sqrt{\xi_1^+\xi_2^+}\left(\bar{\psi}_2\phi_3^\dagger\bar{\psi}_1 + \psi^2\phi^3\psi^1\right)\Bigg)$$

Dilatation Operator and Non-Hermitian Spin Chains

As in ordinary $\mathcal{N}$=4 SYM, the one-loop dilatation operator yields a nearest neighbor spin chain Hamiltonian $\hat{\mathbf{H}}$:

$$\mathfrak{D} = \mathfrak{D}_0 + g^2\,\hat{\mathbf{H}} + \mathcal{O}(g^4)$$

Dropping all fermions, and regarding only chiral composite ops $\mathrm{Tr}\,\phi_{j_1}\phi_{j_2}\phi_{j_3}\ldots$, one gets for $(q_1, q_2, q_3) = (\infty, \infty, \infty)$

$$\hat{\mathbf{H}} = \sum_{\ell=1}^{L} \hat{\mathbb{P}}^{\ell,\ell+1} \quad \text{acting on} \quad \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \cdots \otimes \mathbb{C}^3$$

where the strongly twisted permutation op $\hat{\mathbb{P}}$ acts on sites $\ell, \ell+1$ as

$$\hat{\mathbb{P}}\,|11\rangle = 0 \qquad \hat{\mathbb{P}}\,|22\rangle = 0 \qquad \hat{\mathbb{P}}\,|33\rangle = 0$$

$$\hat{\mathbb{P}}\,|12\rangle = 0 \qquad \hat{\mathbb{P}}\,|23\rangle = 0 \qquad \hat{\mathbb{P}}\,|31\rangle = 0$$

$$\hat{\mathbb{P}}\,|21\rangle = \xi_3^+\,|12\rangle \qquad \hat{\mathbb{P}}\,|32\rangle = \xi_1^+\,|23\rangle \qquad \hat{\mathbb{P}}\,|13\rangle = \xi_2^+\,|31\rangle$$

The Hypereclectic Spin Chain

Specializing to $\xi_1^+ = \xi_2^+ = 0$, $\xi_3^+ = 1$ one gets the hypereclectic model:

$$\mathfrak{H} = \sum_{\ell=1}^{L} \mathfrak{P}^{\ell,\ell+1} \quad \text{acting on} \quad \underbrace{\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \cdots \otimes \mathbb{C}^3}_{L-\text{ times}}$$

with periodic boundary conditions, and where $\mathfrak{P}$ acts on sites $\ell, \ell+1$ as

$$\begin{aligned}
&\mathfrak{P}\,|11\rangle = 0 &\qquad& \mathfrak{P}\,|22\rangle = 0 &\qquad& \mathfrak{P}\,|33\rangle = 0 \\
&\mathfrak{P}\,|12\rangle = 0 && \mathfrak{P}\,|23\rangle = 0 && \mathfrak{P}\,|31\rangle = 0 \\
&\mathfrak{P}\,|21\rangle = |12\rangle && \mathfrak{P}\,|32\rangle = 0 && \mathfrak{P}\,|13\rangle = 0\,.
\end{aligned}$$

Could there be a simpler spin chain Hamiltonian?

As we shall see, this model is integrable, but the standard quantum inverse scattering method fails. Interestingly, its "spectrum" is more complicated than the one of the eclectic model with "generic" $\xi_1^+, \xi_2^+, \xi_3^+$.

Integrability of the Eclectic Spin Chain, I

The R-matrix of the eclectic model reads

$$\hat{\mathbf{R}}(u) = \left(\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & & & 1 & & & & & \\
 & & \xi_2^+ u & & & & 1 & & \\
\hline
 & 1 & & \xi_3^+ u & & & & & \\
 & & & & 1 & & & & \\
 & & & & & & & 1 & \\
\hline
 & & 1 & & & & & & \\
 & & & & & 1 & & \xi_1^+ u & \\
 & & & & & & & & 1
\end{array}\right)$$

It satisfies the Yang-Baxter equation:

$$\hat{\mathbf{R}}^{12}(u-u')\hat{\mathbf{R}}^{13}(u)\hat{\mathbf{R}}^{23}(u') = \hat{\mathbf{R}}^{23}(u')\hat{\mathbf{R}}^{13}(u)\hat{\mathbf{R}}^{12}(u-u')$$

Integrability of the Eclectic Spin Chain, II

In standard fashion, the quantum monodromy matrix is then built as

$$\hat{\mathbf{M}}^{a,L}(u) = \hat{\mathbf{R}}^{a,L}(u) \cdot \hat{\mathbf{R}}^{a,L-1}(u) \cdot \ldots \cdot \hat{\mathbf{R}}^{a,2}(u) \cdot \hat{\mathbf{R}}^{a,1}(u)$$

Also satisfies the YBE. The transfer matrix is $\hat{\mathbf{T}}(u) := \mathrm{Tr}_a \, \hat{\mathbf{M}}(u)$, while

$$\hat{\mathbf{H}} = \mathbf{U}^{-1} \frac{d}{du} \hat{\mathbf{T}}(u) \Big|_{u=0} \quad \text{with the shift operator} \quad \mathbf{U} = \hat{\mathbf{T}}(0)$$

It thus encodes a tower of commuting charges, including the Hamiltonian:

$$[\hat{\mathbf{T}}(u), \hat{\mathbf{T}}(u')] = 0 \quad \text{and hence} \quad [\hat{\mathbf{H}}, \hat{\mathbf{T}}(u')] = 0$$

This renders the eclectic spin chain integrable by two of the possible definitions of quantum integrability: Quantum YBE and charges in involution.

Integrability of the Hypereclectic Spin Chain

The R-matrix of the hypereclectic model reads

$$
\mathfrak{R}(u) = \left(\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & & & 1 & & & & & \\
 & & & & & & 1 & & \\
\hline
 & 1 & & u & & & & & \\
 & & & & 1 & & & & \\
 & & & & & & & 1 & \\
\hline
 & & 1 & & & & & & \\
 & & & & & 1 & & & \\
 & & & & & & & & 1
\end{array}\right)
$$

Being just the special case $\xi_1^+ = \xi_2^+ = 0$, $\xi_3^+ = 1$ of the eclectic model, the above construction works in the very same way. This proves the model's quantum integrability. But is it also exactly solvable?

Non-Diagonalizability of the (Hyper)eclectic Model

For hermitian Hamiltonians $\mathbf{H}$, we know that there must be $j = 1, \ldots, 3^L$ linearly independent eigenstates $|\psi_j\rangle$ satisfying, with $\omega_L := e^{\frac{2\pi i}{L}}$,

$$\mathbf{H}\,|\psi_j\rangle = E^j\,|\psi_j\rangle \quad \text{where} \quad \mathbf{U}\,|\psi_j\rangle = \omega_L^{k_j}\,|\psi_j\rangle \quad \text{and} \quad k_j \in \{0, \ldots, L-1\}$$

For the eclectic model, the eigenvalue equation has to be replaced by

$$\left(\hat{\mathbf{H}} - E^j\right)^{m_j}\,|\psi_j^{m_j}\rangle = 0 \quad \text{with} \quad m_j = 1, \ldots, l_j$$

The $|\psi_j^{m_j}\rangle$ are generalized eigenstates with generalized eigenvalues E^j.

Note that the Hamiltonian $\hat{\mathbf{H}}$ is still block-diagonal w.r.t. sectors of fixed numbers $L-M$ of fields ϕ_1, $M-K$ fields ϕ_2, and K fields ϕ_3. And the $|\psi_j^{m_j}\rangle$ may still be chosen to be eigenstates of $\mathbf{U}$ with eigenvalues $\omega_L^{k_j}$.

Relation to Logarithmic Conformal Field Theory

On the field theory side, Jordan blocks in the spectral problem lead to logarithms in correlation functions. For example

$$\mathfrak{D}\begin{pmatrix}\mathcal{O}_1\\ \mathcal{O}_2\end{pmatrix} = \begin{pmatrix}\Delta & 1\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}\mathcal{O}_1\\ \mathcal{O}_2\end{pmatrix} \rightarrow \langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle \sim \frac{1}{|x|^{2\Delta}}\begin{pmatrix}\log x^2 & 1\\ 1 & 0\end{pmatrix}.$$

Logarithmic conformal field theories in two dimensions have been a major topic in mathematical physics in the last 20 years or so. They are often mathematically more subtle than ordinary CFTs. In two dimensions, there are very important applications to statistical mechanics systems such as polymers. There the non-unitarity is not a problem, and actually beneficial.

Chiral XY-Model

For $K = 0$ (no fields ϕ_3) the non-hermitian Hamiltonian is actually diagonalizable, either by Bethe ansatz, or else a Jordan-Wigner transformation:

$$E = \sum_{m=1}^{M} \frac{1}{u_m^-} \quad \text{and} \quad \omega_L^k = \prod_{m=1}^{M} \frac{1}{\xi_3^+ \, u_m^-}$$

where, in the sector of M fields ϕ_2, one has

$$(\xi_3 \, u_m^-)^L = 1 \quad \text{for} \quad m = 1, \ldots M$$

One easily checks the completeness of all the $\binom{L}{M}$ states of this sector. This clearly leads to the completeness of all 2^L states with $K = 0$.

But what happens for $K > 1$, i.e. the three-state model?

Jordan Normal Form

For $K \neq 0$ the Hamiltonian $\hat{\mathbf{H}}$ is not diagonalizable. It turns out that all generalized eigenvalues are $E = 0$. Define $l \times l$ Jordan blocks by

$$J_l := \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix}.$$

The best one can do is to bring $\hat{\mathbf{H}}$ into Jordan Normal Form (JNF) by a similarity transform S, composed of b blocks of sizes l_j:

$$S \cdot \hat{\mathbf{H}} \cdot S^{-1} = \begin{pmatrix} J_{l_1} & & 0 \\ & \ddots & \\ 0 & & J_{l_b} \end{pmatrix} := l_1 l_2 \ldots l_b \quad \text{with} \quad l_1 + \ldots + l_b = 3^L - 2^L$$

Example: Hypereclectic $L = 3, M = 2, K = 1$

Work in 'cyclic sector', where all states are invariant under the shift operator $\hat{T}(0)$. For $L = 3, M = 2, K = 1$ there are 2 states

$$|123\rangle + |312\rangle + |231\rangle\,, \quad |213\rangle + |321\rangle + |132\rangle\,,$$

which we write simply as

$$|123\rangle_c\,, \quad |213\rangle_c\,.$$

We clearly identify single a Jordan block of size 2

$$|213\rangle_c \xrightarrow{H} |123\rangle_c \xrightarrow{H} 0,$$

$$H^{\mathsf{cyc}}_{3,2,1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Example: Hypereclectic General L, $M = 2$, $K = 1$

Things are also trivial for higher L, $M = 2, K = 1$. There is a single Jordan block of size $L - 1$

$$|211\ldots13\rangle \to |121\ldots13\rangle \to |112\ldots13\rangle \to \cdots \to |111\ldots23\rangle \to 0,$$

Things become much more intricate for higher M and K!

(Hyper)eclectic JNF Spectrum: $M = 2, 3$ and $K = 1$

Thus one easily proves that for $M = 2$, $K = 1$ one has $\text{JNF}_L = (L-1)^L$.
For $M = 3$, $K = 1$ one finds, first numerically, for the cyclic sector

L	Size of sector	JNF
5	6	1 5
7	15	1 5 9
9	28	1 5 9 13
6	10	3 7
8	21	3 7 11
10	36	3 7 11 15

This led us to the conjecture

$$\text{JNF}_L = \prod_{j=1}^{\left[\frac{L-1}{2}\right]} (2L - 4j - 1)$$

Hyperclectic JNF Spectrum: $M = 4$ and $K = 1$

L	Sizes of Jordan Blocks
6	3 7
10	3 7^2 9 11 13 15 19
14	3 7^2 9 11^2 13^2 15^2 17 19^2 21 23 25 27 31
8	1 5 7 9 13
12	1 5 7 9^2 11 13^2 15 17 19 21 25
16	1 5 7 9^2 11 13^3 15^2 17^2 19^2 21^2 23 25^2 27 29 31 33 37
7	4 6 10
11	4 6 8 10^2 12 14 16 18 22
15	4 6 8 10^2 12^2 14^2 16^2 18^2 20 22^2 24 26 28 30 34
9	4 6 8 10 12 16
13	4 6 8 10^2 12^2 14 16^2 18 20 22 24 28
17	4 6 8 10^2 12^2 14^2 16^3 18^2 20^2 22^2 24^2 26 28^2 30 32 34 36 40

Based on this, we conjectured the following recursion (cyclic states):

$$\mathsf{JNF}_{L+4} = \mathsf{JNF}_L\,(L+1)(L+3)(L+5)\ldots(L+(2L+1))$$

Hyperclectic JNF Spectrum: $M = 5$ and $K = 1$

L	Sizes of Jordan Blocks
8	$1 \; 5 \; 7 \; 9 \; 13$
9	$1 \; 5^2 \; 9^2 \; 11 \; 13 \; 17$
10	$1 \; 5^2 \; 7 \; 9^2 \; 11 \; 13^2 \; 15 \; 17 \; 21$
11	$1^2 \; 5^2 \; 7 \; 9^3 \; 11 \; 13^3 \; 15 \; 17^2 \; 19 \; 21 \; 25$
12	$1 \; 5^3 \; 7 \; 9^3 \; 11^2 \; 13^3 \; 15^2 \; 17^3 \; 19 \; 21^2 \; 23 \; 25 \; 29$
13	$1^2 \; 5^3 \; 7 \; 9^4 \; 11^2 \; 13^4 \; 15^2 \; 17^4 \; 19^2 \; 21^3 \; 23 \; 25^2 \; 27 \; 29 \; 33$
14	$1^2 \; 5^3 \; 7^2 \; 9^4 \; 11^2 \; 13^5 \; 15^3 \; 17^4 \; 19^3 \; 21^4 \; 23^2 \; 25^3 \; 27 \; 29^2 \; 31 \; 33 \; 37$
15	$1^2 \; 5^4 \; 7 \; 9^5 \; 11^3 \; 13^5 \; 15^3 \; 17^6 \; 19^3 \; 21^5 \; 23^3 \; 25^4 \; 27^2 \; 29^3 \; 31 \; 33^2 \; 35 \; 37 \; 41$
16	$1^2 \; 5^4 \; 7^2 \; 9^5 \; 11^3 \; 13^6 \; 15^4 \; 17^6 \; 19^4 \; 21^6 \; 23^4 \; 25^5 \; 27^3 \; 29^4 \; 31^2 \; 33^3 \; 35 \; 37^2 \; 39 \; 41 \; 45$
17	$1^3 \; 5^4 \; 7^2 \; 9^6 \; 11^3 \; 13^7 \; 15^4 \; 17^7 \; 19^5 \; 21^7 \; 23^4 \; 25^7 \; 27^4 \; 29^5 \; 31^3 \; 33^4 \; 35^2 \; 37^3 \; 39 \; 41^2 \; 43 \; 45 \; 49$
18	$1^2 \; 5^5 \; 7^2 \; 9^6 \; 11^4 \; 13^7 \; 15^5 \; 17^8 \; 19^5 \; 21^8 \; 23^6 \; 25^7 \; 27^5 \; 29^7 \; 31^4 \; 33^5 \; 35^3 \; 37^4 \; 39^2 \; 41^3 \; 43 \; 45^2 \; 47 \; 49 \; 53$

Based on this, we conjectured the following recursion (cyclic states):

$$\mathsf{JNF}_{L+5} = \frac{\mathsf{JNF}_{L+2}\,\mathsf{JNF}_{L+3}}{\mathsf{JNF}_L}\,(2L+1)(2L+3)\ldots(2L+(2L+1))$$

Question

So how to derive and prove all this numerical data?

- Incidentally, at $K > 2$ things are getting even more intricate!
- Can we use integrability? As I will now sketch, the Bethe ansatz fails!
- On the other hand, a combination of combinatorial and linear algebra methods leads to success.

Bethe Ansatz: Intricate, but Failing, I

Integrable spin chains are usually solved by Bethe ansatz. Applying it directly to the eclectic spin chain, it algebraically fails. Before taking $\varepsilon \to 0$ in the twisted model with $q_j = \varepsilon^{-1}\xi_j^+$ it works perfectly:

$$E = \varepsilon L + \varepsilon \sum_{m=1}^{M} \left(\frac{1}{u_m} - \frac{1}{u_m + 1} \right)$$

with the Bethe equations ($\xi := \xi_1^+ \xi_2^+ \xi_3^+$)

$$\left(\frac{u_m + 1}{u_m} \right)^L = \varepsilon^{3K-L} \, \frac{\xi_3^{+L}}{\xi^K} \prod_{\substack{j=1 \\ j \neq m}}^{M} \frac{u_m - u_j + 1}{u_m - u_j - 1} \prod_{i=1}^{K} \frac{u_m - v_i - 1}{u_m - v_i}$$

$$1 = \varepsilon^{3M-2L} \, \frac{\xi^{L-M}}{\xi_1^{+L}} \prod_{j=1}^{M} \frac{v_l - u_j + 1}{v_l - u_j} \prod_{\substack{i=1 \\ i \neq l}}^{K} \frac{v_l - v_i - 1}{v_l - v_i + 1}$$

Clearly very singular. Still, their limit may in most cases be analyzed.

Bethe Ansatz: Intricate, but Failing, II

E.g. for a rather generic (L, M, K) sector with $L > 3(M-K)$ fractional scaling solutions maybe found explicitly:

$$\begin{aligned} &(\mathrm{I}) \qquad & u_j = \varepsilon^{\alpha}\, u_j^- \,, \qquad & j = 1, \cdots, M-K \\ &(\mathrm{II}) \qquad & u_{l+M'} = -1 + \varepsilon^{\beta}\, u_l^+ \,, \qquad & l = 1, \cdots, K \\ &(\mathrm{III}) \qquad & v_l = -2 + \varepsilon^{\beta}\, u_l^+ + \varepsilon^{\gamma}\, \hat{v}_l, \qquad & l = 1, \cdots, K \end{aligned}$$

On may explicitly find the scaled roots $u_j^-, u_l^+, \hat{v}_l$ and the exponents

$$\alpha = \frac{L-(M+K)}{L-(M-K)} \qquad \beta = \frac{L-3(M-K)}{L-(M-K)}$$

$$\gamma = 2L - 3M - \frac{L-3(M-K)}{L-(M-K)}\,(K-1)$$

Proves $E = 0$. But all Bethe states collapse to a trivial "locked" state: $|\phi_1 \ldots \phi_1\, \phi_2 \ldots \phi_2 \ldots \phi_3 \ldots \phi_3\rangle := |1 \ldots 1\, 2 \ldots 2\, 3 \ldots 3\rangle$. JNF ???

Combinatorial Ansatz: $L = 7, M = 3, K = 1$

There are 15 states. Ansatz for 'top state' for a Jordan block of size 9:

$$
\begin{aligned}
&|2211113\rangle && H^0 \\
&\rightarrow |2121113\rangle && H^1 \\
&\rightarrow |2112113\rangle + |1221113\rangle && H^2 \\
&\rightarrow |2111213\rangle + 2\,|1212113\rangle && H^3 \\
&\rightarrow |2111123\rangle + 3\,|1211213\rangle + 2\,|1122113\rangle && H^4 \\
&\rightarrow 4\,|1211123\rangle + 5\,|1121213\rangle && H^5 \\
&\rightarrow 5\,|1112213\rangle + 9\,|1121123\rangle && H^6 \\
&\rightarrow 14\,|1112123\rangle && H^7 \\
&\rightarrow 14\,|1111223\rangle && H^8 \\
&\rightarrow 0 && H^9
\end{aligned}
$$

$L = 7, M = 3, K = 1$, continued

However, the state space is still not exhausted: 6 states are missing. Ansatz for second top state: $\alpha\,|2112113\rangle + \beta\,|1221113\rangle$.

$$
\begin{aligned}
&\alpha\,|2112113\rangle + \beta\,|1221113\rangle \\
&\to \beta\,|2111213\rangle + (\alpha+\beta)\,|1212113\rangle \\
&\to \beta\,|2111123\rangle + (\alpha+2\beta)\,|1211213\rangle + (\alpha+\beta)\,|1122113\rangle \\
&\to (\alpha+3\beta)\,|1211123\rangle + (2\alpha+3\beta)\,|1121213\rangle \\
&\to (2\alpha+3\beta)\,|1112213\rangle + (3\alpha+6\beta)\,|1121123\rangle \\
&\to (5\alpha+9\beta)\,|1112123\rangle = 0
\end{aligned}
$$

if $\alpha = -9, \beta = 5$. This determines a Jordan block of length 5.

However, the state space is $still$ not exhausted: 1 state ist missing!

$L = 7, M = 3, K = 1$, completed, and an Insight

Ansatz for a final, top=bottom state:

$\alpha' |2111123\rangle + \beta' |1211213\rangle + \gamma' |1122113\rangle$. This is an eigenstate for $\alpha' = -\beta' = \gamma' = 1$, giving a Jordan block of length 1.

Conclusion:
The JNF in the cyclic sector for $L = 7, M = 3, K = 1$ is $(9, 5, 1)$.

Insight:
The full Jordan block structure can be deduced by computing $\dim(H^k |\phi\rangle)$.

q-Combinatorics

Encode this structure in a partition function

$$Z_{7,3}(q) = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8.$$

Problem is solved if we can calculate $Z_{L,M}(q)$. It turns out that it is a q-Binomial coefficient

$$Z_{L,M}(q) = \binom{L-1}{M-1}_q = \prod_{k=1}^{M-1} \frac{1-q^{L-k}}{1-q^k},$$

which are always polynomials.

The q^k coefficients of these have a combinatorial interpretation as counting the number of partitions of the integer k subject to certain restrictions.

More Examples

For $M = 2$ the partition functions are very simple

$$Z_{7,2}(q) = 1 + q + q^2 + q^3 + q^4 + q^5.$$

This indicates a single block of size 6.

It neatly encodes the involved structures in the previous tables for $K = 1$. For example

$$\begin{aligned} Z_{9,5}(q) =& 1 + q + 2q^2 + 3q^3 + 5q^4 + 5q^5 + 7q^6 + 7q^7 + 8q^8 \\ & + 7q^9 + 7q^{10} + 5q^{11} + 5q^{12} + 3q^{13} + 2q^{14} + q^{15} + q^{16} \end{aligned}$$

leads to a Jordan block spectrum $(17, 13, 11, 9^2, 5^2, 1)$.

General Situation (To Appear)

One may rewrite the partition function as

$$Z(q) = \mathsf{tr}\, q^{\hat{S}},$$

for an appropriate state-counting operator $\hat{S}$.
Generalizes naturally to arbitrary K!

Eclectic: Universality hypothesis: Spectrum of hypereclectic matches that of the eclectic provided the filling conditions

$$L - M \geq M - K \geq K$$

are satisfied.

Conclusions

- Inspired by strongly twisted $\mathcal{N}$=4 SYM, we considered novel classes of non-diagonalizable spin chains: The Eclectic and Hypereclectic models.

- We proved their quantum integrability by deriving their R-matrices.

- We showed that the Bethe ansatz equations make sense, and can even be partially solved explicitly, exhibiting rather non-trivial scaling behavior. However, vexingly, they appear to be utterly useless for determining the "spectrum" of Jordan Normal Forms.

- With a combination of linear algebra and combinatorial methods, we are able to determine the JNF of the Hypereclectic model. The solution is encoded in a suitable partition function. The resulting spectrum is intricate and non-trivial. Integrability is not (directly) used.

To Do

- Complete the exact solution of the Hypereclectic model.
- Prove the universality hypothesis for the Eclectic model.
- Understand how to use the integrability of (Hyper)eclectic model.
- Derive the consequences of the JNF on strongly twisted $\mathcal{N}=4$ SYM. Should be very non-trivial examples of four-dimensional non-unitary logarithmic quantum field theories.
- Non-perturbative solutions via the quantum spectrum curve (QSC) have been proposed, largely ignoring the JNF structure. Implications?
- Higher loops, strong coupling, and dual "Fish Chain"? [N. Gromov, A. Sever '19]