On Factorizable S-matrices, Generalized TTbar, and the Hagedorn Transition

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Outline

- Motivations and Introduction
- Integrability and the generalized TTbar deformations
- Thermodynamic Bethe Ansatz
- Results for many models: primary and secondary branch
- Conclusions and open questions

Introduction and Motivation

(Smirnov, Zamolodchikov)

(Dubovsky,Gorbenko,Mirbabayi)

TTbar proper deformation: one-parameter family of formal "actions" $\mathcal{A}_\alpha$

$$\frac{d}{d\alpha}\mathcal{A}_\alpha = \int (T\bar{T})_\alpha(x)\, d^2x \,, \qquad T\bar{T} \propto \det T_{\mu\nu}$$

"irrelevant" operator

Solvable deformation

S-matrix: $S_\alpha(\theta) = S_0(\theta)\, \exp\left(-i\alpha M^2 \sinh\theta\right)$ rapid growth of the $2 \to 2$ scattering phase.

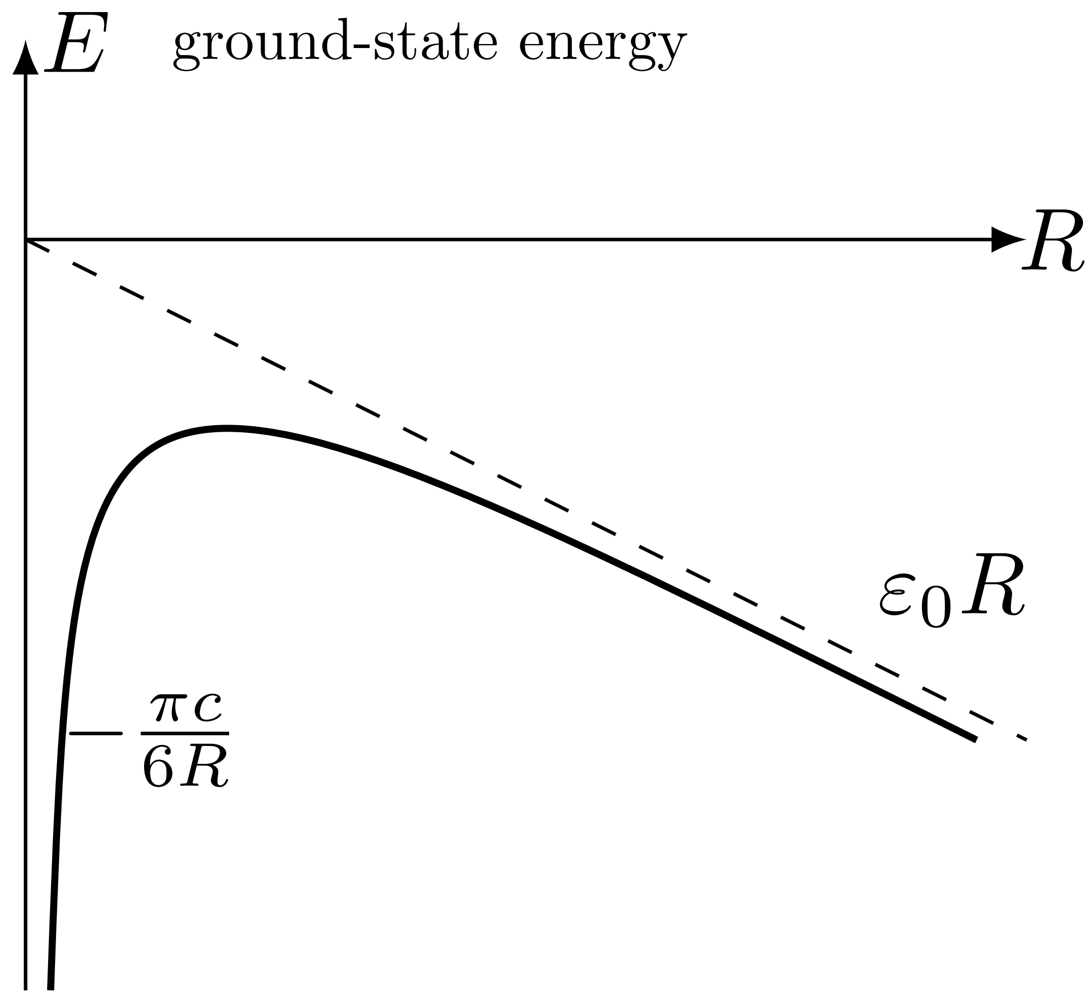
Remaining of the talk: $M = 1$

Are there other deformations with a qualitative similar behaviour?

Is the behaviour of the scattering phase a necessary condition?

Conventional Relativistic QFT

spatial coordinate of the 2D space-time is compactified on a circle of circumference R



ε_0

Bulk vacuum density

c Central Charge

TTbar proper

(Smirnov, Zamolodchikov)

(Cavaglià, Negro, Szécsényi,Tateo)

Ground State Energy $$E_\alpha(R) = E_0(R - \alpha E_\alpha(R))$$

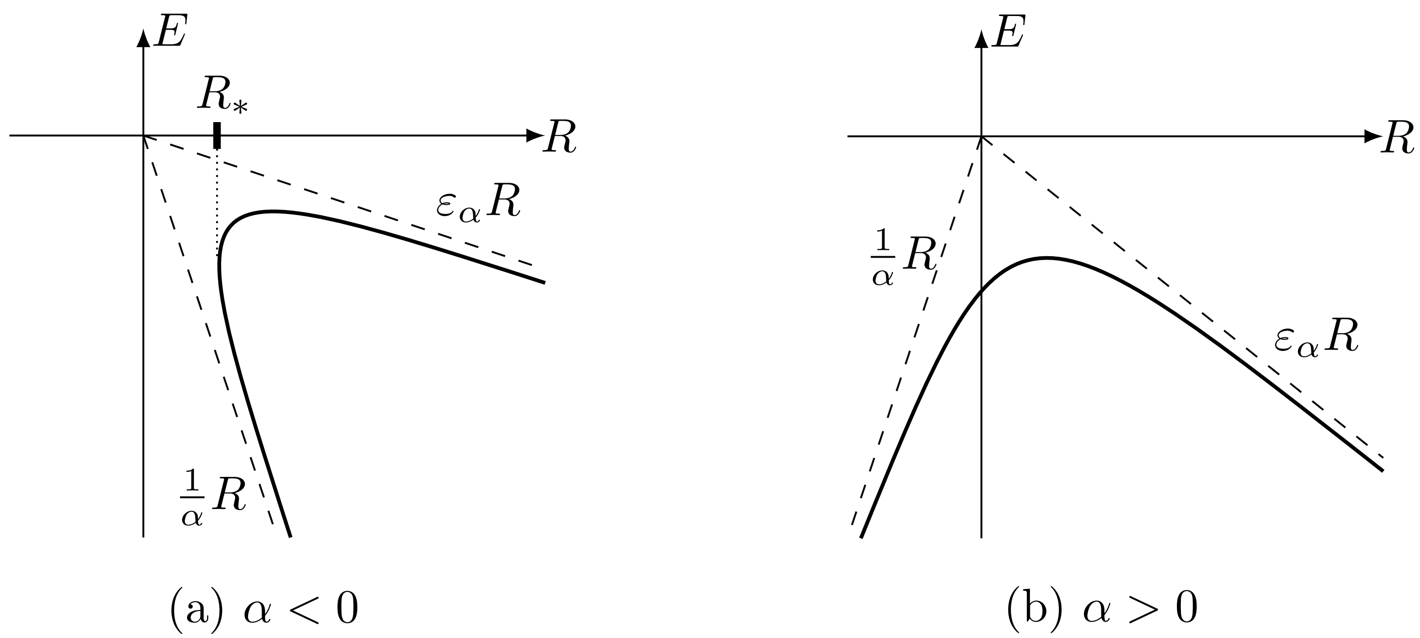


Figure 2: Finite-size ground state energy of the TTbar deformed theory. (a) $\alpha < 0$. The graph $E_\alpha(R)$ shows the "turning point" at some finite R_*, which signals the Hagedorn transition. (b) $\alpha > 0$. $E_\alpha(R)$ shows no singularity at $R = 0$.

This talk: $\boxed{\text{negative } \alpha}$ Two branches:

Primary branch or "physical"

Secondary branch

Generalized TTbar and Integrable systems

(Smirnov, Zamolodchikov)

conserved local currents of higher Lorentz spins $s+1$, pairs of local fields $(T_{s+1}(z), \Theta_{s-1}(z))$

$$\partial_{\bar{\mathrm{z}}} T_{s+1}(z) = \partial_{\mathrm{z}} \Theta_{s-1}(z)$$

Sine-Gordon: $\{s\}$ of odd natural numbers: $s = 1, 3, 5, 7, \ldots$

$$\lim_{z \to z'} (T_{s+1}(z)\bar{T}_{s-1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')) = T\bar{T}^{(s)}(z') + \text{derivatives}$$

Notation: for negative s, i.e. for $s > 0$ we write Θ_{-s-1} as $\bar{T}_{s+1}$ and T_{-s+1} as $\bar{\Theta}_{s-1}$

$$\partial_{\mathrm{z}} \bar{T}_{s+1}(z) = \partial_{\bar{\mathrm{z}}} \bar{\Theta}_{s-1}(z)$$

Family of deformations:

$$\frac{\partial \mathcal{A}_{\{\alpha\}}}{\partial \alpha_s} = \int T\bar{T}^{(s)}_{\{\alpha\}}(x)\, d^2x \ .$$

CDD Factors

following deformation of the elastic two-particle S-matrix

$$S_{\{\alpha\}}(\theta) = S_{\{0\}}(\theta)\,\Phi_{\{\alpha\}}(\theta)\,, \qquad \Phi_{\{\alpha\}}(\theta) = \exp\left\{-i\sum_{s\in 2\mathbb{Z}+1} \alpha_s\,\sinh(s\,\theta)\right\}$$

$\Phi_{\{\alpha\}}(\theta)$ is CDD factor

unitarity and crossing $\qquad \Phi(\theta)\Phi(-\theta) = 1\,, \qquad \Phi(\theta) = \Phi(i\pi - \theta)$

Different basis

$$\Phi_{\text{pole}}(\theta) = \prod_{p=1}^{N} \frac{\sinh\theta_p + \sinh\theta}{\sinh\theta_p - \sinh\theta}\,,$$

$$\Phi_{\{\alpha\}}(\theta) = \Phi_{\text{pole}}(\theta)\,\Phi_{\text{entire}}(\theta)\,,$$

$$\Phi_{\text{entire}}(\theta) = \exp\left\{-i\sum_{s} a_s\,\sinh(s\,\theta)\right\}$$

TBA equations: Finite size ground state energy

"bosonic TBA" when $\sigma = +1$ and "fermionic TBA" for $\sigma = -1$

Given $S(\theta)$ let $\varphi(\theta)$

$$\varphi(\theta) = \frac{1}{i} \frac{d}{d\theta} \log S(\theta)$$

$\epsilon(\theta)$ *pseudo-energy*

TBA equations:

$$\epsilon(\theta) = R \cosh\theta - \int \varphi(\theta - \theta')\, L(\theta')\, \frac{d\theta'}{2\pi} \qquad L(\theta) := -\sigma \log\left(1 - \sigma\, e^{-\epsilon(\theta)}\right)$$

Energy:

$$E(R) = -\int_{-\infty}^{\infty} \cosh\theta\, L(\theta)\, \frac{d\theta}{2\pi}\ .$$

TBA and TTbar proper deformations

(Cavaglià, Negro, Szécsényi,Tateo)

(Dubovsky, Flauger, Gorbenko)

(Caselle, Fioravanti, Gliozzi, Tateo)

(LeClair)

Deformed S-matrix: $$S_\alpha(\theta) = S_0(\theta)\,\exp\left(-i\alpha M^2 \sinh\theta\right)$$

Deformed Kernel: $$\varphi_\alpha(\theta-\theta') = \varphi_0(\theta-\theta') - \alpha\,\cosh(\theta-\theta')$$

ground state energy $E_\alpha(R)$

$$E_\alpha(R) = -\int_{-\infty}^{\infty} \cosh\theta\; L_\alpha(\theta|R)\,\frac{d\theta}{2\pi} \qquad L_\alpha(\theta|R) := \log\left(1+e^{-\epsilon_\alpha(\theta|R)}\right)$$

deformed TBA equation

$$\epsilon_\alpha(\theta|R) = R\cosh\theta - \int \varphi_\alpha(\theta-\theta')\,L_\alpha(\theta'|R)\,\frac{d\theta'}{2\pi}$$

Thus:

$$\epsilon_\alpha(\theta|R) = (R-\alpha\,E_\alpha(R))\,\cosh\theta - \int \varphi_0(\theta-\theta')\,L_\alpha(\theta'|R)\,\frac{d\theta'}{2\pi}$$

Deformed Energy:

$$\epsilon_\alpha(\theta|R) = \epsilon_0(\theta|R-\alpha E_\alpha(R))$$

The Models Considered

CDD deformations of the trivial (fermionic or bosonic) S-matrix

$$S(\theta) = \sigma \prod_{p=1}^{N} \frac{i \sin u_p + \sinh\theta}{i \sin u_p - \sinh\theta}$$

$\sigma = -$ (resp. $\sigma = +$) corresponds to the fermionic (resp. bosonic) case

Restrictions: periodicity $-\pi < \mathrm{Re}(u_p) < \pi$

One stable particle: poles $\theta_p = i u_p$ real positive u_p stable particles of mass $2M \cos(u_p/2)$

Not allowed!

analytic requirements

The 1CDD models

(a) $u_1 \in \mathbb{R}$ and $-\pi < u_1 < 0$,

(b) $u_1 = -\pi/2 + i\theta_0$ and $\theta_0 \in \mathbb{R}$.

Fermionic:

case (a) S-matrix of the sinh-Gordon model

$$S_{\rm shG}(\theta) = -\frac{i \sin u_1 + \sinh\theta}{i \sin u_1 - \sinh\theta}$$

case (b) S-matrix of the "staircase model"

$$S_{\rm stair}(\theta) = \frac{\sinh\theta - i\cosh\theta_0}{\sinh\theta + i\cosh\theta_0}\ , \qquad \theta_0 \in \mathbb{R}$$

Bosonic version: Mussardo and Simon

The 2CDD model

$$S_{2\mathrm{CDD}}(\theta) = \sigma \, \frac{i \sin u_1 + \sinh\theta}{i \sin u_1 - \sinh\theta} \frac{i \sin u_2 + \sinh\theta}{i \sin u_2 - \sinh\theta}$$

(a) $u_1 \in \mathbb{R}$ and $-\pi < u_1 < 0$,
$u_2 \in \mathbb{R}$ and $-\pi < u_2 < 0$,

(b) $\theta_0 \in \mathbb{R}$ and $u_1 = -\pi/2 + i\theta_0$,
$u_2 \in \mathbb{R}$ and $-\pi < u_2 < 0$,

(b') $u_1 \in \mathbb{R}$ and $-\pi < u_1 < 0$,
$\theta_0 \in \mathbb{R}$ and $u_2 = -\pi/2 + i\theta_0$,

(c) $\theta_0 \in \mathbb{R}$ and $u_1 = -\pi/2 + i\theta_0$,
$\theta_0' \in \mathbb{R}$ and $u_2 = -\pi/2 + i\theta_0'$,

(d) $\theta_0 \in \mathbb{R}$, $\gamma \in (-\pi/2, \pi/2)$, $u_1 = \gamma - \pi/2 + i\theta_0$ and $u_2 = u_1^*$.

3CDD, 4CDD, ...

Narrow Resonance Limit (NRL)

special limit $\gamma \to \frac{\pi}{2}$ kernel : two Dirac δ functions

TBA becomes the difference equation

$$Y(\theta|R) = e^{-R\cosh\theta}[1-\sigma Y(\theta+\theta_0|R)]^{-\sigma}[1-\sigma Y(\theta-\theta_0|R)]^{-\sigma}$$

where $Y(\theta|R) = e^{-\epsilon(\theta|R)}$ grid points $\theta \in (-\theta_0, \theta_0) + \theta_0\mathbb{Z}$

Notation: fermionic case $(\sigma = -1)$ bosonic case $(\sigma = +1)$

NRL: Fermionic Case

Introducing $y_k = Y(\theta + k\theta_0)$ and $g_k = e^{-R\cosh(\theta + k\theta_0)}$

NRL equations: $y_k = g_k(1 + y_{k-1})(1 + y_{k+1}) \qquad (k \in \mathbb{Z})$

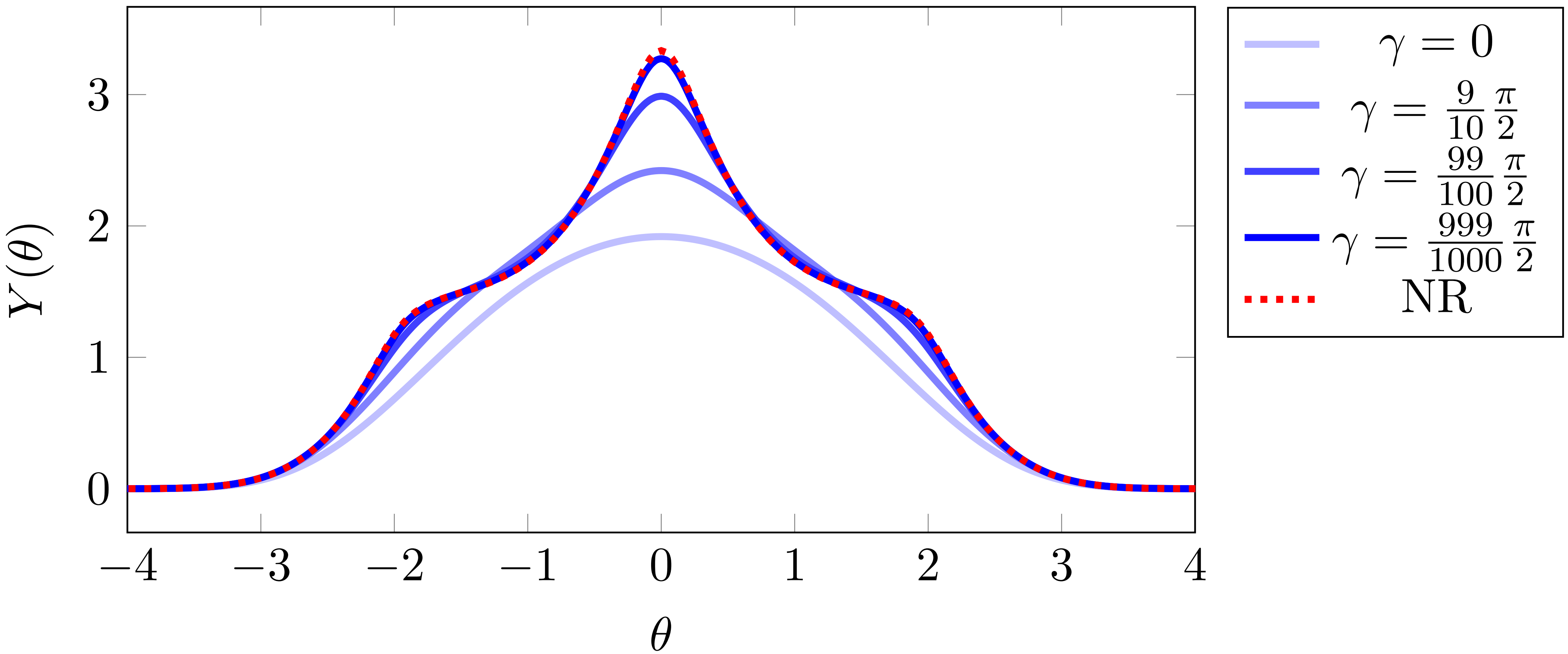
truncating the system for some $|k| \leq m$, since g_k and y_k decay very rapidly with R and θ_0, and hence with k

$$m = 1$$

$$y_0 = -1 - e^{-R\cosh\theta_0} + \frac{1}{2} e^{R(1+2\cosh\theta_0)} \left(1 \pm \sqrt{1 - 4e^{-R(1+\cosh\theta_0)}(1 + e^{-R\cosh\theta_0})}\right).$$

branching point depends on the choice of θ lattice.

$m = 8$



$\theta_0 = 2$ and $x = 1.75$ $x = \log(R/2)$

Case: $\theta_0 = 0$

Simple algebraic equation

TBA: Iterative Solution

(Fring, Korff, Schulz)

The equations can be solved analytically for a very few cases

Iterative Procedure:

$$\epsilon_n(\theta) = R\cosh\theta + \sigma\int \varphi(\theta-\theta')\log\left[1-\sigma e^{-\epsilon_{n-1}(\theta')}\right]\frac{d\theta}{2\pi}$$

$$\lim_{n\to\infty}\epsilon_n(\theta) = \epsilon(\theta)$$ And $$\epsilon_0(\theta) = R\cosh\theta$$

The convergence and uniqueness has been proven rigorously for the fermionic case with

$$||\varphi||_1 := \int |\varphi(\theta)|\frac{d\theta}{2\pi} \leq 1$$

Note: $||\varphi_{N\text{CDD}}||_1 = N$

positive "critical radius" $R_* > 0$ such that for $R \leq R_*$ the iterative routine stops converging

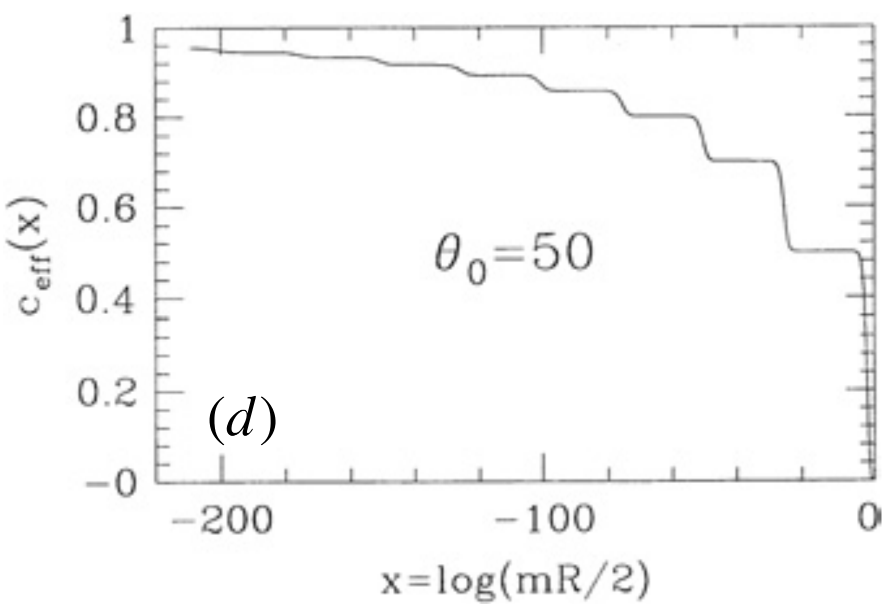
Staircase Model (Al. Zamolodchikov)

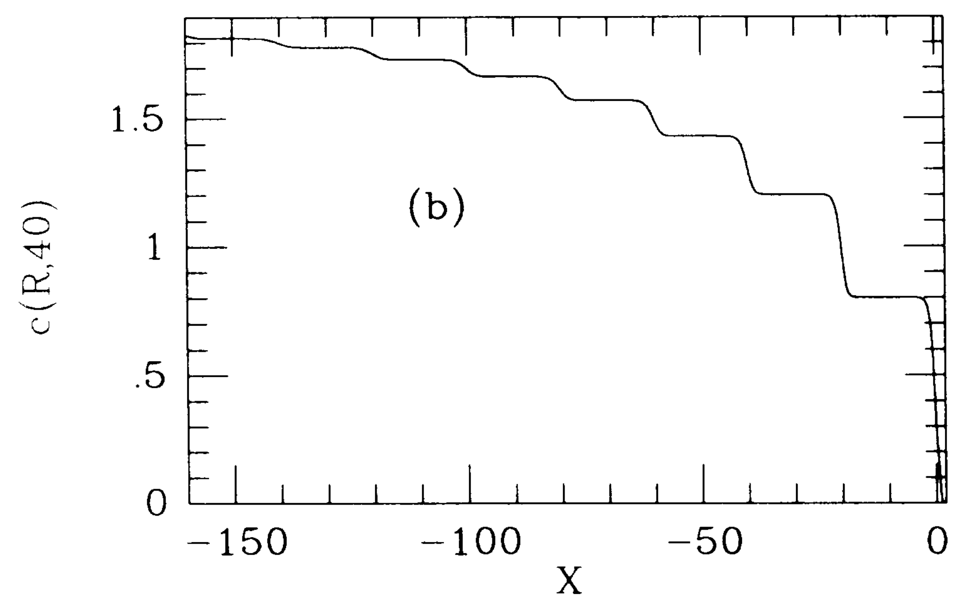
3 CDD's (M. Martins)

$$S(\theta) = \frac{\sinh\theta - \mathrm{i}\cosh\theta_0}{\sinh\theta + \mathrm{i}\cosh\theta_0}.$$

$$u_1 = -\frac{4\pi}{3}$$

$$u_2 = \frac{1}{3}(-2\pi + 3ia) \qquad u_3 = \frac{1}{3}(-2\pi - 3ia)$$





$$c_p = 1 - 6/(p(p+1))$$

$$c = 2[1 - 12/p(p+1)],\ p = 4, 5, \ldots .$$

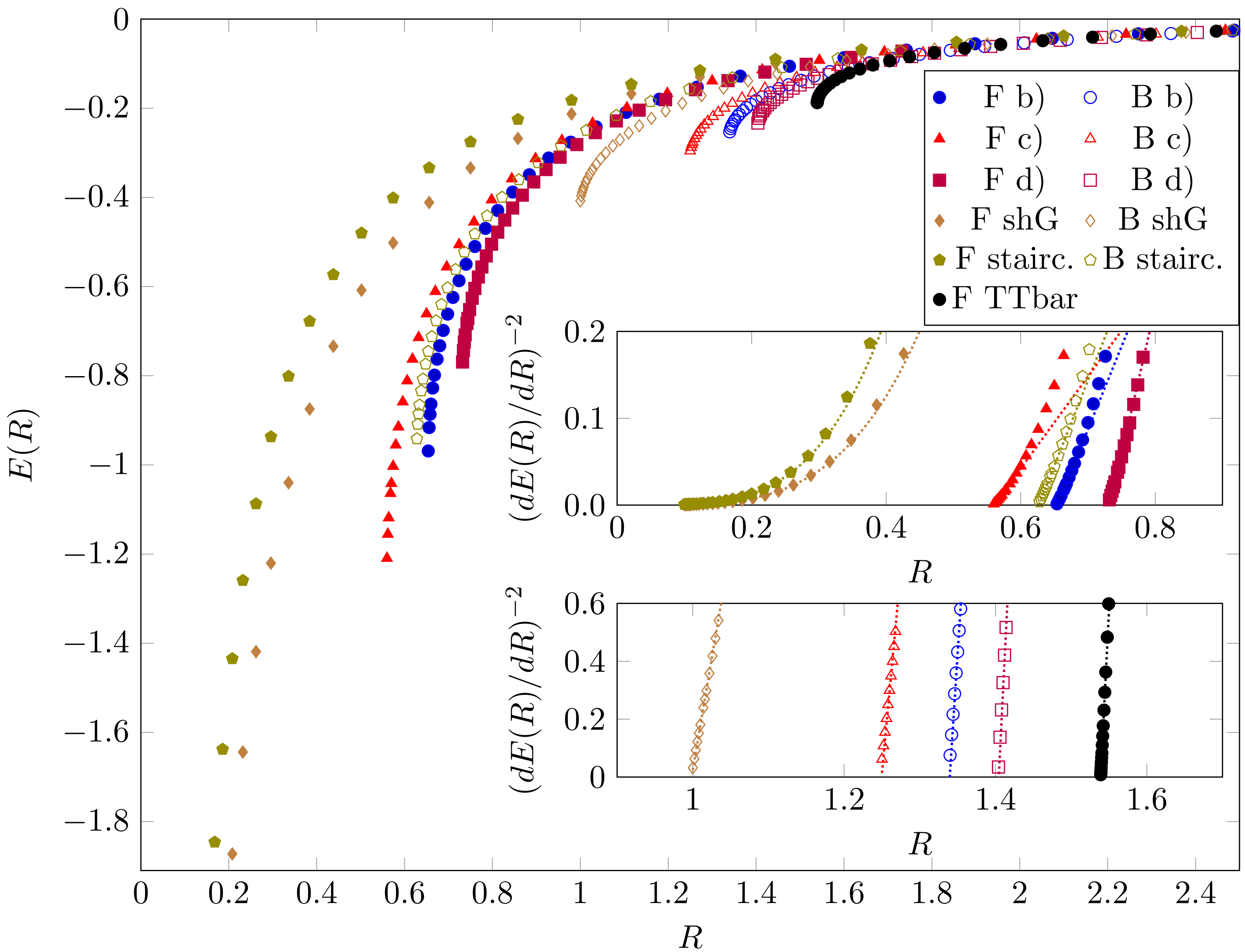
conformal minimal models $\mathcal{M}_p$

the minimal model of the $W(A_2)$

Two Branches

$$E(R) \underset{R\to R_*^+}{\sim} c_0 + c_{1/2}\sqrt{R-R_*} + \mathcal{O}(R-R_*)$$

$$\tilde{E}(R) \underset{R\to R_*^+}{\sim} c_0 - c_{1/2}\sqrt{R-R_*} + \mathcal{O}(R-R_*)$$



Ground-state energies for various models

F: fermionic

B: bosonic

Casimir behaviour: R^4

Square root behavior: R

The pseudo-arc-length continuation method

(Allgower, Georg)

critical points *turning points.* $\frac{d}{dR}C(R)$ diverges (no bifurcations)

truncate and discretize the real θ-line on a N-point lattice $\{\theta_k \mid k = 1, 2, \ldots, N\}$

$$H : \begin{array}{ccc} \mathbb{R}^N \times \mathbb{R} & \longrightarrow & \mathbb{R}^N \\ (\vec{\epsilon}, R) & \longmapsto & \vec{H}(\vec{\epsilon}, R) \end{array} , \qquad H_k(\vec{\epsilon}, R) = -\epsilon_k + R\cosh\theta_k - \frac{1}{2\pi}\sum_l \Delta\theta\varphi_{kl}\log\left(1 + e^{-\epsilon_l}\right)$$

fixed-point condition $\vec{H}(\vec{\epsilon}, R) = \vec{0}$.

Goal: follow a curve: $\vec{H}(C(s)) = \vec{0}$ starting point $C_i = (\vec{\epsilon}_i, R_i)$ final one $C_f = (\vec{\epsilon}_f, R_f)$

Solution: a good parematrization: s arc-length of C

initial value problem $H'(C(s))\dot{C}(s) = \vec{0}$, $||\dot{C}(s)|| = 1$, $C(s_i) = (\vec{\epsilon}_i, R_i)$ $\dot{C}(s) = \begin{pmatrix} \frac{d}{ds}\vec{\epsilon} \\ \hline \frac{d}{ds}R \end{pmatrix}$

Steps

Predictor

$$(\vec{\epsilon}^{(0)}_{j+1}, R^{(0)}_{j+1}) = (\vec{\epsilon}_j, R_j) + \delta s \, \frac{t_j}{||t_j||} \ ,$$

Moving in the curve:

$$H'(\vec{\epsilon}_j, R_j) t_j = 0 \ .$$

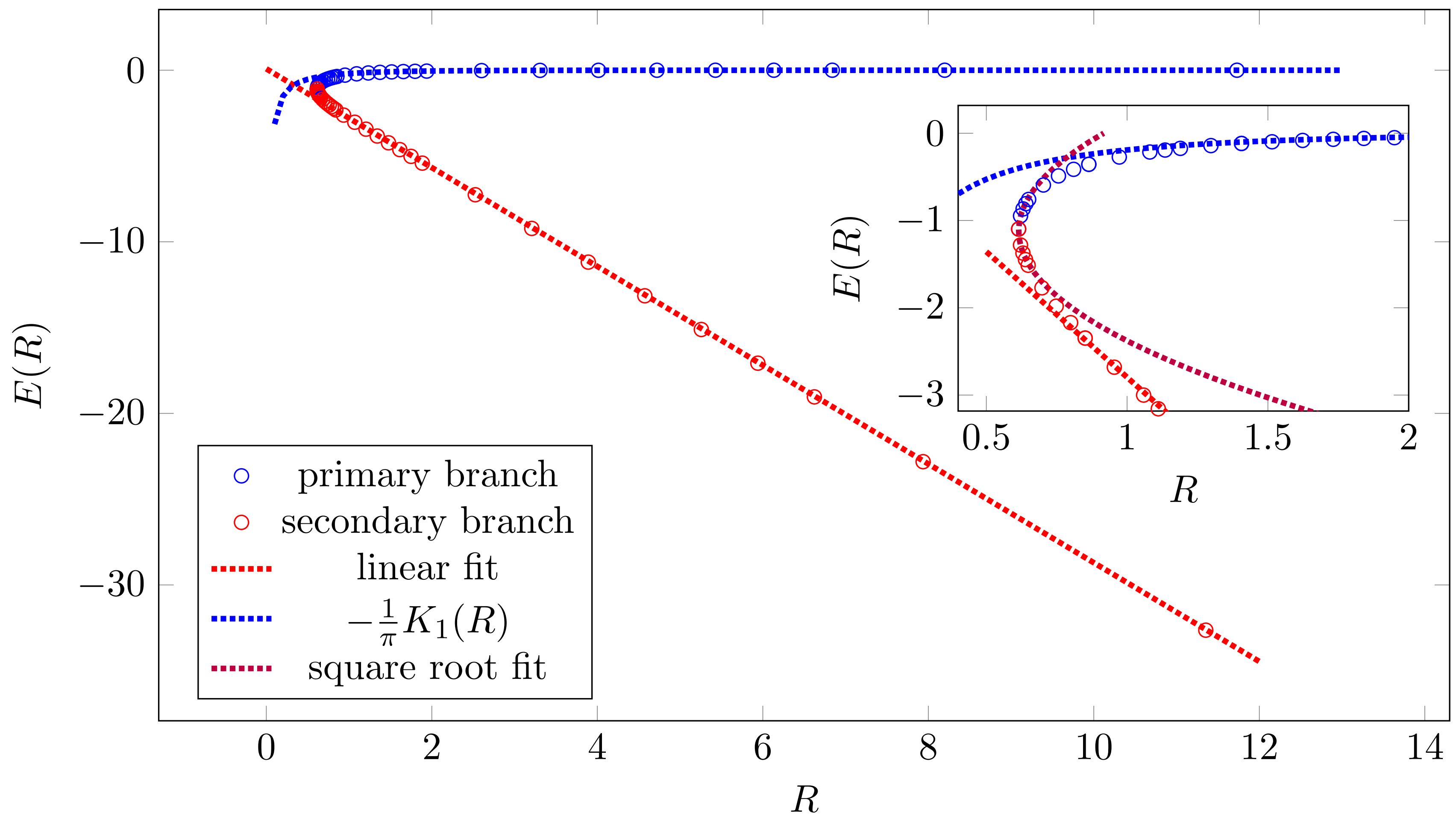
Corrector

point actually lying on the solution curve

Newton's method $N \times (N+1)$

inverse of the Jacobian for the quasi-inverse of the extended Jacobian

Example of the PALC continuation



2CDD Fermionic

$\theta_0 = 1/2$ and $\gamma = 3\pi/20$

primary branch $$E(R) \underset{R\to\infty}{\sim} -\frac{1}{\pi} K_1(R) + \mathcal{O}\left(e^{-2R}\right) \ ,$$

secondary branch $$\tilde{E}(R) \underset{R\to\infty}{\sim} -\varepsilon_- R$$

Results 2CDD model

Some analytical results: large-R behaviors

$$\epsilon(\theta) = d(\theta) - \chi(\theta) \,,$$

$$d(\theta) = R\cosh\theta \,, \qquad \chi(\theta) = \int \varphi(\theta - \theta') \, \log\left[1 + e^{-\epsilon(\theta')}\right] \frac{d\theta'}{2\pi}$$

For: $R \to \infty$ $\quad d(\theta) \sim R$

Condition: either $\epsilon(\theta)$, $\chi(\theta)$ or both $\sim R$

Case 1:

$$\epsilon(\theta) \underset{R\to\infty}{\sim} d(\theta) \,, \qquad \chi(\theta) \underset{R\to\infty}{\ll} d(\theta)$$

$$\chi(\theta) \underset{R\to\infty}{\sim} \int \varphi(\theta - \theta') \, \log\left[1 + e^{-R\cosh\theta'}\right] \frac{d\theta'}{2\pi} \underset{R\to\infty}{\sim} \frac{\varphi(\theta)}{\sqrt{2\pi R}} e^{-R} \underset{R\to\infty}{\ll} R\cosh\theta$$

Case 2:

$$\epsilon(\theta) \underset{R\to\infty}{\sim} -R\, f(\theta)\ , \qquad \begin{cases} f(\theta) > 0\ , & \theta \in \Theta \subset \mathbb{R}\ , \\ f(\theta) \leq 0 & \theta \in \Theta^{\perp} = \mathbb{R} - \Theta \end{cases}$$

and $\quad \chi(\theta) \sim \epsilon(\theta)$

Possible if there is a solution to: $$f(\theta) = -\cosh\theta + \int_{\Theta} \varphi(\theta - \theta')\, f(\theta')\, \frac{d\theta'}{2\pi}\ .$$

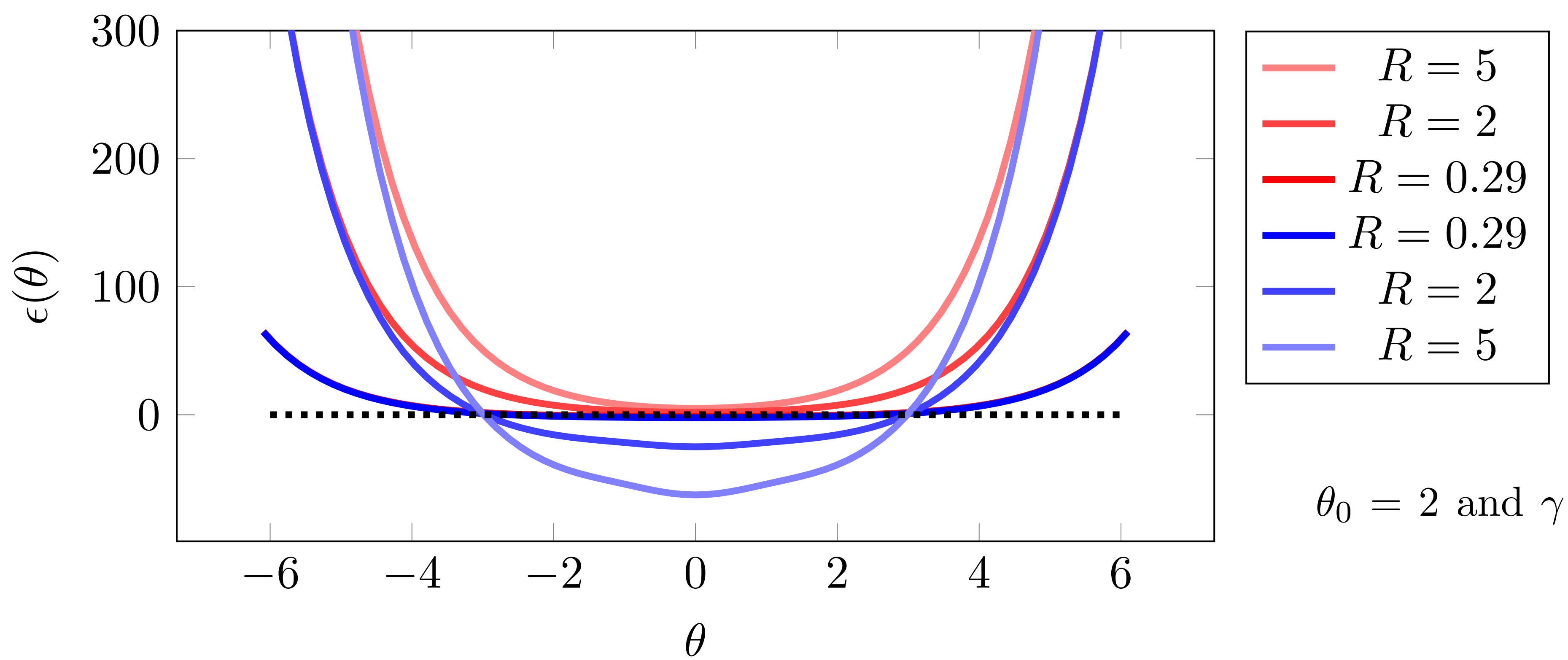
Necessary condition:

$$N > \int_{\Theta} \varphi(\theta_{\mathrm{M}} - \theta')\, \frac{d\theta'}{2\pi} > 1 \quad \Longrightarrow \quad N > 1$$

where $\quad \theta_{\mathrm{M}} \in \Theta \qquad f(\theta_{\mathrm{M}}) = \underset{t\in\Theta}{\mathrm{Max}}\,[f(t)]$

Numerical Results

representative case (d)



$\theta_0 = 2$ and $\gamma = 4\pi/10$

$\epsilon(\theta)$ for the secondary branch solution (blue) , iterative solution (red)

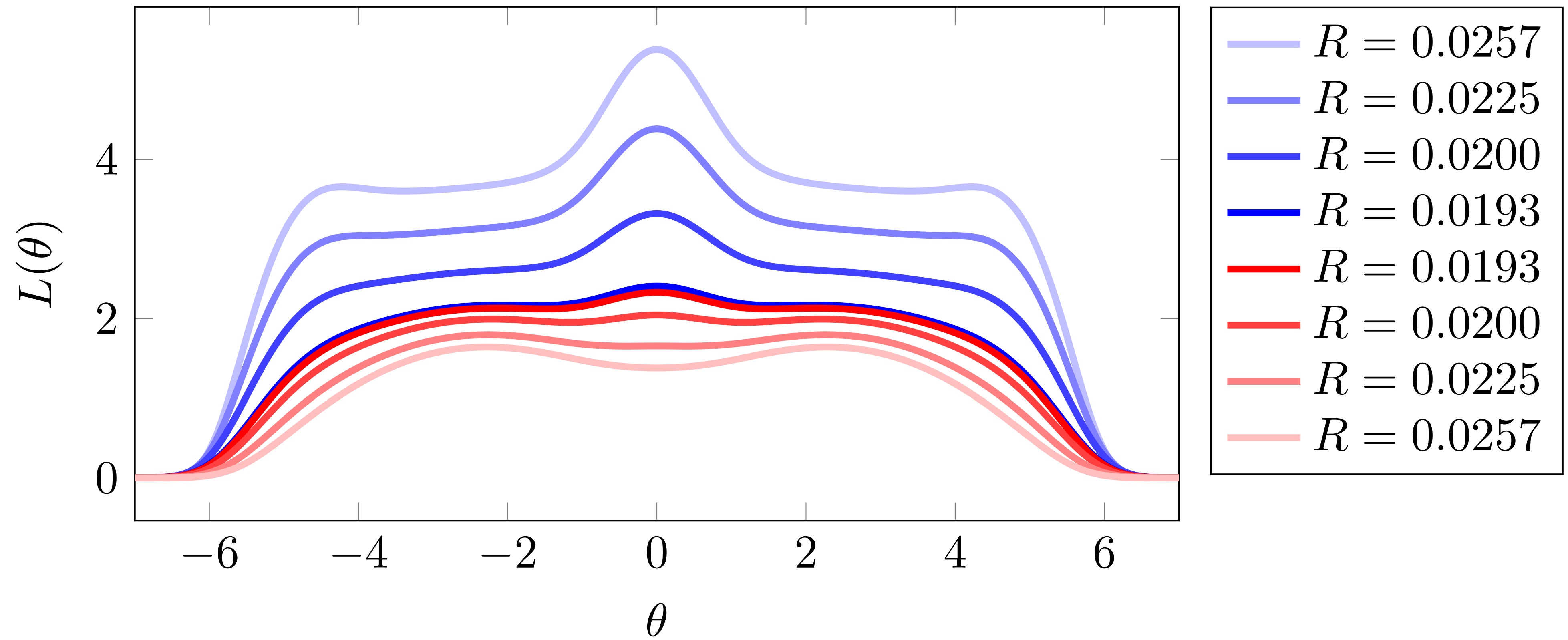
grow with θ_0 and decreases

Just one interval: $\Theta = \{\theta \in \mathbb{R} | -\Lambda \leq \theta \leq \Lambda\}$

with γ

Cont.

Fermionic: R approaches the critical value R_*



$\theta_0 = 5$

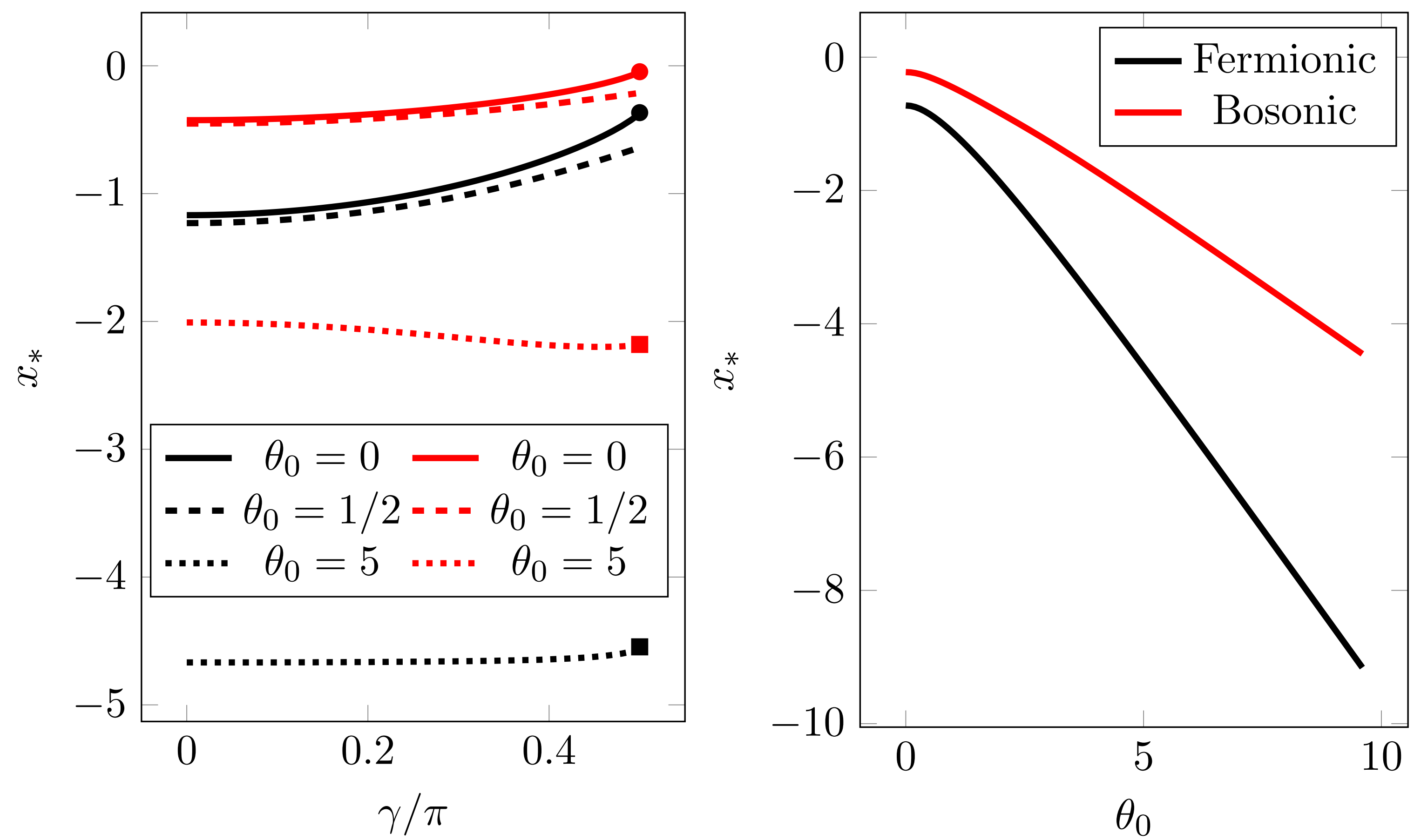
$\gamma = 4\pi/10$

$R_* \approx 0.0192$

$L(\theta)$ for both the primary (red) and secondary (blue)

Recall: $L(\theta) := -\sigma \log\left(1 - \sigma\, e^{-\epsilon(\theta)}\right)$

Cont.



(a) γ dependence of x_*.

(b) θ_0 dependence of x_* with $\gamma = 2\pi/5$

Non-Linear Fitting: $a(\theta) + b(\theta)\sqrt{-x_*(\theta) + x}.$ Independent of θ

they agree up to errors greater than our minimal working precision

several values of θ_0 and for γ in the range $0 \leq \gamma \leq (99/200)\pi$

Bosonic Case

PALC method jumps back to the iterative solution

pair of complex conjugate zeros of $z(\theta) = 1 - e^{-\epsilon(\theta)}$ is approaching the real axis

Solution: map between fermionic and bosonic TBA equations

If: $\epsilon(\theta)$ is a solution of the bosonic TBA equation

$\tilde{\epsilon}(\theta) = \log\left[e^{\epsilon(\theta)} - 1\right]$ is a solution of the fermionic TBA equation

with kernel $\tilde{\varphi}(\theta) = \varphi(\theta) + 2\pi\delta(\theta)$

Conclusion:

bosonic NCDD model

is equivalent to the $(N+1)$CDD fermionic TBA

taken in the limit when $u_{N+1} \to 0$

Conclusions and Open Problems

- Physics of the secondary branch

- For $R < R_*$ complex energy

- Physical conditions for formation of the singularity

- Analytical proofs: square root behaviour, independence on theta, size of the negative interval, more CDDs

- CDD's with entire part as well; massless cases

Thank you very much!

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