$T\bar{T}$ deformation of Witten's cigar

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Content

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Liouville equation

$$\partial\bar{\partial}\,\Phi(z,\bar{z}) + \mu^2\, e^{2\Phi(z,\bar{z})} = 0$$

The general solution

$$\Phi(z,\bar{z}) = \frac{1}{2}\ \log\ \frac{A'(z)\,\bar{A}'(\bar{z})}{[1\,+\mu^2\,A(z)\bar{A}(\bar{z})]^2}$$

$z = \tau + \sigma$, $\bar{z} = \tau - \sigma$ are chiral coordinates.

WZW field $\mathfrak{g}(z,\bar{z}) = g(z)\,\bar{g}(\bar{z})$, with g and $\bar{g} \in G$.

Imposing constraints on the Kac-Moddy currents

$$J(z) = g'(z)\,g^{-1}(z) \qquad \bar{J}(\bar{z}) = \bar{g}^{-1}(\bar{z})\,\bar{g}'(\bar{z})$$

one gets a coset WZW model.

The gauge invariant components of $\mathfrak{g}$ become the fields of the coset model.

In Liouville theory the parameterising chiral fields are simply related to the asymtotic in or out fields, which are massless free-fields

$$\Phi_{\mathsf{in}}(z,\bar{z}) = \phi_{\mathsf{in}}(z) + \bar{\phi}_{\mathsf{in}}(\bar{z})$$

$$A'(z) = \mathsf{e}^{2\phi_{\mathsf{in}}(z)} \qquad \bar{A}'(\bar{z}) = \mathsf{e}^{2\bar{\phi}_{\mathsf{in}}(\bar{z})}$$

Similar relations hold for the out fields and one relates the aymptotic fields

$$\mathsf{e}^{-\phi_{\mathsf{out}}(z)} = \mu \mathsf{e}^{-\phi_{\mathsf{in}}(z)} A(z)$$

From this relation follows

$$\phi'^{\,2}_{\mathsf{out}}(z) - \phi''_{\mathsf{out}}(z) = \phi'^{\,2}_{\mathsf{in}}(z) - \phi''_{\mathsf{in}}(z)$$

which is interpreted as $T_{\mathsf{out}}(z) = T_{\mathsf{in}}(z) = T(z)$, where

$$T(z) = (\partial\Phi)^2 - \partial^2\Phi$$

We study Liouville theory on a cylinder with $\sigma \in S^1$.
The asymptotic fields obey the mode expansion

$$\phi_{\mathsf{in}}(z) = q + \frac{p\,z}{2} + \mathsf{i}\sum_{n\neq 0} \frac{a_n}{n}\,\mathsf{e}^{-\mathsf{i}nz} \qquad \phi_{\mathsf{out}}(z) = \tilde{q} - \frac{p\,z}{2} + \mathsf{i}\sum_{n\neq 0} \frac{b_n}{n}\,\mathsf{e}^{-\mathsf{i}nz}$$

with $p > 0$.

After canonical quantization we introduce the asymptotic coherent states

$$|p, a\rangle_{\mathsf{in}} = \exp\left(\frac{2}{\hbar}\sum_{m>0}\frac{1}{m}\,a_m\,\hat{a}_m^\dagger\right)|p, 0\rangle_{\mathsf{in}}\ .$$

The out-field 'bra' vectors ${}_{\mathsf{out}}\langle b^*, \tilde{p}|$, with $\tilde{p} < 0$, are given similarly.

Our aim is to calculate the S-matrix

$$ {}_{\mathsf{out}}\langle b^*, \tilde{p} | p, a \rangle_{\mathsf{in}} = \mathcal{S}(p, b^*, a)\, \delta(p + \tilde{p}) $$

It has the structure

$$ \mathcal{S}(p, b^*, a) = R(p)\, \tilde{\mathcal{S}}(p, j_n) $$

with

$$ \tilde{\mathcal{S}}(p, j_n) = 1 + \sum_{\nu \geq 2} \frac{1}{\nu!} \sum_{n_1, \dots n_\nu} S_{n_1, \dots n_\nu}(p)\, j_{n_1} \dots j_{n_\nu}\, \delta_{n_1 + \dots + n_\nu} $$

Here $\{b_m^*, a_m; m > 0\} \sim \{j_n, n \neq 0\}$, and $R(p)$ is the reflection amplitude (the 2-point function)

$$ R(p) = -\left(\mu^2 \frac{\sin(\pi \hbar)}{\pi \hbar} \Gamma^2(\hbar) \right)^{-\frac{ip}{\hbar}} \frac{\Gamma\left(ip/\hbar\right)}{\Gamma\left(-ip/\hbar\right)} \frac{\Gamma(ip)}{\Gamma(-ip)} $$

Low level transition amplitudes

$$S_{-1\,1} = i\,\frac{1+\hbar-ip}{1+\hbar+ip}\ ,$$

$$S_{-1\,-1\,2} = \frac{-4ip(1+\hbar)}{3(1+\hbar+ip)(1+2\hbar+ip)(2+\hbar+ip)} = -S_{-2\,1\,1}$$

$$S_{-1\,-1\,1\,1} = \frac{-4p(1+\hbar)}{3(1+\hbar+ip)^2(1+2\hbar+ip)(2+\hbar+ip)}$$

$$S_{-2\,2} = \frac{i(2+7\hbar+7\hbar^2+2\hbar^3)-(3+5\hbar+3\hbar^2)p-p^3}{2(1+\hbar+ip)(1+2\hbar+ip)(2+\hbar+ip)}$$

The Lagrangian of Witten's cigar

$$\mathcal{L} = \frac{1}{2} \; \frac{\partial u \, \bar{\partial} u^* + \bar{\partial} u \, \partial u^*}{1 + u u^*}$$

where u is a complex coordinate on a plane.

The target space metric can be written in polar coordinates

$$\frac{\mathsf{d}u \, \mathsf{d}u^*}{1 + u u^*} = \frac{(\mathsf{d}R)^2}{(1 - R^2)^2} + R^2 \, (\mathsf{d}\theta)^2$$

which corresponds to the induced metric on the rotational surface in $\mathbb{R}^3$:
$X_1 = R \cos\theta, \;\; X_2 = R \sin\theta, \;\; X_3 = f(R),$ with f obtained from

$$1 + f'^{\,2} = \frac{1}{(1 - R^2)^2}$$

and it provides an infinite 'cigar' in $\mathbb{R}^3$.

Witten introduced it in 1991, as the $\mathsf{SL}(2,\mathbb{R})/\mathsf{U}(1)$ coset WZW model.

The dynamical equation

$$\bar{\partial}\partial u = u^* \, \frac{\partial u \, \bar{\partial} u}{1 + u u^*} \; .$$

provides that chiral are the stress tensor components

$$T(z) = \frac{\partial u \, \partial u^*}{1 + u u^*} \qquad \bar{T}(\bar{z}) = \frac{\bar{\partial} u \, \bar{\partial} u^*}{1 + u u^*}$$

and also the local fields

$$W(z) = \frac{\partial^2 u}{\partial u} - \frac{u \, \partial u^*}{1 + u u^*} \qquad \bar{W}(\bar{z}) = \frac{\bar{\partial}^2 u}{\bar{\partial} u} - \frac{u \, \bar{\partial} u^*}{1 + u u^*}$$

The field u then satisfies the linear equations

$$\partial^2 u = W(z) \, \partial u + T(z) \, u \qquad \bar{\partial}^2 u = \bar{W}(\bar{z}) \, \bar{\partial} u + \bar{T}(\bar{z}) \, u$$

Liouville theory:

Liouville field in terms of asymptotic fields

$$\mathsf{e}^{-\Phi(z,\bar{z})} = \mathsf{e}^{-\Phi_{\mathsf{in}}(z,\bar{z})} + \mathsf{e}^{-\Phi_{\mathsf{out}}(z,\bar{z})}$$

The relation between the asymptotic fields

$$\mathsf{e}^{-\Phi_{\mathsf{out}}(z,\bar{z})} = \mathsf{e}^{-\Phi_{\mathsf{in}}(z,\bar{z})} A(z) \bar{A}(\bar{z})$$

$$A'(z) = \mathsf{e}^{2\phi_{\mathsf{in}}(z)} \qquad \bar{A}'(\bar{z}) = \mathsf{e}^{2\bar{\phi}_{\mathsf{in}}(\bar{z})}$$

Witten's cigar

$$u(z,\bar{z}) = \mathsf{e}^{-\Phi_{\mathsf{in}}(z,\bar{z})} + \mathsf{e}^{-\Phi_{\mathsf{out}}(z,\bar{z})}$$

$$\mathsf{e}^{-\Phi_{\mathsf{out}}(z,\bar{z})} = \mathsf{e}^{-\Phi_{\mathsf{in}}(z,\bar{z})} A(z) \bar{A}(\bar{z})$$

Φ_{in} is a complex free-field $\Phi_{\mathsf{in}}(z,\bar{z}) = \Phi_1(z,\bar{z}) + \mathsf{i}\Phi_2(z,\bar{z})$

$$A'(z) = \mathsf{i}\left[\phi_1'(z) + \mathsf{i}\phi_2'(z)\right] \mathsf{e}^{2\phi_1(z)}$$

In quantum Liouville theory the stress tensor gets a correction

$$T =: \phi'^2 : -(1+\hbar)\phi''$$

and the Virasoro generators, in terms of the in-field variables, are

$$\hat{L}_0 = \frac{1}{4}p^2 + 2\sum_{j\geq 1} \hat{a}_{-j}\hat{a}_j$$

$$\hat{L}_m = \left[\hat{p} + i\,m\,(1+\hbar)\right]\hat{a}_m + \sum_{i,j\neq 0} \hat{a}_i\hat{a}_j\delta_{i+j,m}$$

The same generators in terms of the out-field variables are obtained by the replacements $p \mapsto -p$, $\hat{a}_j \mapsto \hat{b}_j$.

Projecting the operator identity $T_{\mathsf{out}}(z) = T_{\mathsf{in}}(z)$ between the asymptotic coherent states

$$\langle \tilde{p}, b^* | T_{\mathsf{out}}(z) | p, a \rangle = \langle \tilde{p}, b^* | T_{\mathsf{in}}(z) | p, a \rangle$$

and taking its Fourier modes, one gets linear equations for $\mathcal{S}(p, b^*, a)$, which are simplified after the rescaling

$$\mathcal{S} = e^{-\frac{1}{4\hbar}\sum_{n\neq 0}\frac{j_n j_{-n}}{|n|}}\tilde{\mathcal{S}}_p(j)$$

Inserting the Laplace transform of $\tilde{\mathcal{S}}_p(j)$ in the simplified equation, one represents it as a functional integral, where the integrand is otained from the equation.

One gets

$$\tilde{\mathcal{S}}_p[j] = \frac{Z_p[j]}{Z_p[0]} = 1 + \sum_{\nu \geq 2} \frac{b^\nu}{\nu!} \sum_{n_1,\ldots,n_\nu} S_p^{\,n_1\ldots n_\nu}\, j_{n_1} \cdots j_{n_\nu}\, \delta_{n_1+\cdots+n_\nu}$$

where $b = \sqrt{\hbar}$, $j_n \to b j_n$ $\varphi_n \to \varphi_n$ and

$$Z_p(j) = \int \prod_{n\neq 0} d\varphi_n \, \exp\left[\sum_{n\neq 0}\left(\varphi_{-n}\, j_n - \frac{1}{2}\,\varphi_{-n}|n|\varphi_n\right)\right] A^{-\frac{ip}{b^2}}$$

with

$$A = \int \frac{dx}{2\pi} \exp\left(b \sum_{n\neq 0} \varphi_n\, e^{-inx}\right)$$

The aim is to calculate $Z_p(j)$ and find the expansion coefficients $S_p^{n_1\ldots n_\nu}$.

We first analyze $Z_p(j)$ for $p = Nib^2$, where N is a positive integer. In this case one can calculate the functional integral and represent it as

$$Z_{Nib^2}(j) = \exp\left(\frac{1}{2}\sum_{n\neq 0}\frac{j_{-n}\, j_n}{|n|}\right) I_N(j)$$

where

$$I_N(j) = \int \prod_{\alpha=1}^{N} \frac{dx_\alpha}{2\pi} \exp\left[\frac{1}{2}\sum_{n\neq 0}\frac{1}{|n|}\left(2b\, j_{-n}\sum_{\alpha=1}^{N} e^{inx_\alpha} + b^2 \sum_{\alpha,\beta=1}^{N} e^{in(x_\alpha - x_\beta)}\right)\right.$$

We neglect the divergent term

$$\frac{b^2}{2}\sum_{n\neq 0}\frac{N}{|n|}$$

that corresponds to $\alpha = \beta$.

Taking into accout the relation

$$\sum_{n\neq 0} \frac{1}{|n|}\, e^{inx} = -\log\left(2 - e^{ix} - e^{-ix}\right),$$

and using the integration variables $z_\alpha = e^{ix_\alpha}$, we rewrite $I_N(j)$ as

$$I_N(j) = \frac{1}{(2\pi i)^N} \oint \prod_{\alpha=1}^{N} \frac{dz_\alpha}{z_\alpha} \exp\left[b \sum_{n\neq 0} \frac{j_{-n}}{|n|}\left(z_1^n + \cdots + z_N^n\right)\right] \times$$
$$\prod_{\alpha\neq\beta}\left(2 - \frac{z_\alpha}{z_\beta} - \frac{z_\beta}{z_\alpha}\right)^{-b^2}.$$

To discuss the $T\bar{T}$ deformation of a 2d CFT, it is convinient to introduce the string model on

$$\mathbb{R} \times S^1 \times \mathcal{M}$$

where $\mathcal{M}$ is the target space of the 2d CFT.

The string model in the light cone gauge is identified with the corresponding 2d CFT and in the static gauge it reproduces its $T\bar{T}$ deformed system.

This relates the deformed system and the initial one by a worldsheet coordinate transformation, which becomes a time dependent canonical map in the Hamiltonian treatment.

The deformed Hamiltonian defines the string energy and it is expressed it in terms of the chiral Hamiltonians of the initial 2d CFT.

This allows exact quantization of the deformed system, if the spectrum of the initial 2d CFT is known.

The deformed Hamiltonian for a sigma-model

$$\mathcal{H}_\alpha = \frac{1}{\alpha}\left(\sqrt{1 + 2\,\alpha\,\mathcal{H} + \alpha^2\,\mathcal{P}^2} - 1\right) \ .$$

One has the map from the coordinates $(z, \bar{z})$ to (τ, σ)

$$\begin{aligned} \tau &= \frac{1}{2}\left[\rho^+ z + \Phi^-(z) + \bar{\rho}^{\,+}\bar{z} + \bar{\Phi}^{\,-}(\bar{z})\right] \ , \\ \sigma &= \frac{1}{2}\left[\rho^+ z - \Phi^-(z) + \bar{\rho}^{\,+}\bar{z} - \bar{\Phi}^{\,-}(\bar{z})\right] \ , \end{aligned} \tag{1}$$

where

$$\Phi'^-(z) = \frac{\alpha}{\rho^+}\,T(z) \qquad \bar{\Phi}'^-(\bar{z}) = \frac{\alpha}{\bar{\rho}^+}\,\bar{T}(\bar{z})$$

The deformed and undeformed fields are rlated by

$$\phi(\tau, \sigma) = u(z, \bar{z}) \ . \tag{2}$$

We have described the symmetries of Wittens cigar, emphasizing its similarity to Liouville theory.

We have introduce the S-matrix of the coset models as an operator with conformal and W-symmetry.

We have discussed the $T\bar{T}$ deformation of the S-matrix.